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Epsilon-Regular Sets and Intervals

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Abstract

Regularity of sets (both open and closed) is fundamental in the classical theory of solid modeling and is implicit in many shape modeling representations. However, strictly speaking, the notion of regularity cannot be applied to real world shapes and/or computed geometric models that usually exhibit irregularity in the forms or errors, uncertainty, and/or approximation. We propose a notion of ε-regularity that quantifies regularity of shapes in terms of set intervals and subsumes the classical notions of open and closed regular sets as special exact cases. Our formulation relies on ε-topological operations that are related to, but are distinct from, the common morphological operations. We also show that ε-regular interval is bounded by two sets, such that the Hausdorff distance between the sets, as well the Hausdorff distance between their boundaries, is at most ε. Many applications of ε-regularity include geometric data translation and solid model validation.

1 Introduction

Regularity of sets is fundamental in the classical theory of solid modeling and is implicit in many shape modeling representations. Both open set and closed set have been used:

- closed regular [24], in which case $X = ki(X)$;
- open regular [3], in which case $X = ik(X)$;

where k, i denote respectively the topological closure and interior operations of a set. Both models correspond to dimensionally homogeneous (without cracks or dangling pieces) sets with tight boundaries, the main difference being that closed regular sets include their boundaries while open regular sets do not. A common characterization in both models is that a neighborhood of every boundary point contains points in the set interior as well as points in its exterior. Specifically, it is widely accepted that a suitable model for an "ideal" solid is an r-set [24], defined as bounded, semi-analytic, and closed regular subset of E^3. The intuitive notion that every non-trivial solid has a non-empty interior and a thin boundary is formally captured by requiring that $X = ki(X)$. With this terminology, a geometric representation is deemed *valid* if it corresponds to at least one r-set, and two representations are *consistent* if they represent the same set of points. The purpose of exact representation conversions is to produce representations that are valid and consistent in accordance with this theory.

By requiring the regularity of sets, the classical solid models postulate that a solid is a set of points with dimensionally homogeneous interior and well defined boundary, where dangling pieces and cracks on the set boundary are not allowed. However, strictly speaking, the notion of regularity does not apply to most real world shapes and/or computed geometric models that usually exhibit irregularity in the forms or errors, uncertainty, and/or approximation. Two common situations where regularity is notoriously difficult to maintain are Boolean set operations and geometric data translation of boundary representations. Theoretically, boundary evaluation and merging algorithms [26] implement the regularized Boolean operations (union, intersection, difference) [25], but finite precision arithmetics and approximations often mean that the output of such procedures does not bound any regular sets. Similarly, after geometric data translation [13, 11, 16, 21, 10], boundary representations of translated solids are often found to have various

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errors, and the receiving systems cannot perform the intended tasks on these models unless they are repaired/healed [6, 5, 8, 7, 20, 30, 29, 32].

Thus, identifying a regular set in the presence of errors is not always feasible, practical, or even desirable. Representation of X is usually generated from user inputs, numerical computations, approximations, and physical samplings – all of which contain some *errors*. These data errors in the representation of X usually violate the validity conditions (numerical, topological, combinatorial) implied by the properties of regular sets. For example, Figure 1 shows typical 'invalid' 2-dimensional boundary representations of a simple rectangle. As with most boundary representations, the vertices, edges, and faces represent geometric information only approximately and redundantly. Depending on how the boundary is constructed, it may also include small imperfections: isolated and dangling pieces, voids, gaps.

Applying directly the notions of (exact) closure, interior, and boundary to such representations and requiring regularity is problematic, because strictly speaking, all such representations are invalid. Either such representations need to be 'repaired' or they must be *reinterpreted* based on semantics that recognizes the role of errors and imprecision. Furthermore, any reasonable repair procedure would also require a prior reinterpretation. One possible model for new semantics of imprecise representations studies conditions under which closed regular sets may be associated with perturbed representations and computations [1, 15]. We advocate an alternative approach that modifies the very notion of regularity, in order to account explicitly for the size ε of errors in geometric models and algorithms. Hence, the key question is: *What is a proper model of regularity* that would tolerate errors of size less than ε, such as those shown in Fig. 1? To answer this question, we proposed the notions of ε-topological operations and ε-regularity that subsume the corresponding classical notions [23] and reinterpret valid geometric representations in terms of *classes* of sets. In this paper, we further explore the properties of ε-regular sets and their applications. In particular, we establish relationship between ε-topological operations and morphological operations, and show that an ε-regular interval is bounded by two sets that are within ε Hausdorff distance of each other. Furthermore, the boundaries of these sets are also at most ε Hausdorff distance apart.

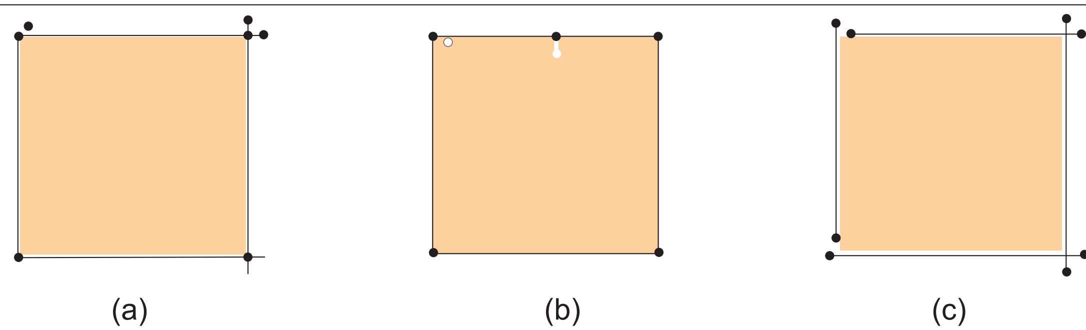

Figure 1: A theory of ε-regularity must tolerate imperfections of size less than ε near the theoretical boundary of a set: (a) dangling edges or isolated points; (b) small cracks and voids; (c) misaligned or redundant vertices and edges.

The paper is organized as follows. Section 2 summarizes the notions of ε-topological operations, ε-regular sets, and ε-regular intervals. Most of the results in this section are also discussed in the recent technical report by the authors [23]. Connection to the morphological operations and characterization of ε-regular intervals in terms of Hausdorff distance properties are studied in Section 3. The last Section 4 discusses the significance of our findings and explains how they can be applied in practice to a number of shape modeling problems, including geometric data translation.

2 ε-Regularity of Sets and Intervals

In this section, we propose an ε-regularity model that captures small irregularities in the forms of errors, uncertainty, and/or approximation. The proposed model quantifies regularity of shapes using ε-topological operations which are generalizations of the corresponding classical set topological notions of *interior* i, *closure* k, and *boundary* ∂. The generalized operations are defined in terms of finite size neighborhoods, to deal with inexactness of data and algorithms. The size of a neighborhood ε relates to the precision of data and algorithms; when precision is limited, the size must be finite. The resulting generalized epsilon-topological operations i_ε, k_ε, and ∂_ε satisfy many (but not all) of the classical topological properties.

Based on the ε-topological operations, we propose notions of ε-regular sets and intervals that explicitly recognize the role of imprecision. Informally, the difference between the classical notions of closed/open regular sets and the ε-regular sets lies in that the set equality is replaced by a pair of set inequalities and the classical topological operations are replaced by the ε-counterparts. The classical open and closed regular sets are shown to be special exact cases where $\varepsilon = 0$ corresponds to arbitrarily small neighborhoods.

2.1 ε-Topological Operations

Traditionally, open and closed regular sets are formulated in terms of the classical topological concepts and operations of closure k, interior i, and boundary ∂ that are interpreted in terms of infinitesimal size neighborhoods. In contrast, inaccuracy of data and finite resolution of algorithms imply that the neighborhoods of every point may be represented only up to some finite size. This in turn requires redefining the usual topological operations.

Specifically, we propose the ε-topological counterparts of the classical topological operations: *ε-closure* k_ε, *ε-interior* i_ε, and *ε-boundary* ∂_ε, where ε is a non-negative real number. Intuitively, ε corresponds to the maximum algorithm precision and/or maximal data precision. In the following definitions, we refer to a metric space as (W, d), where W is a non-empty set and d is a suitable distance function. In a metric space, $B(x, r)$ denotes the open ball about point x of radius r.

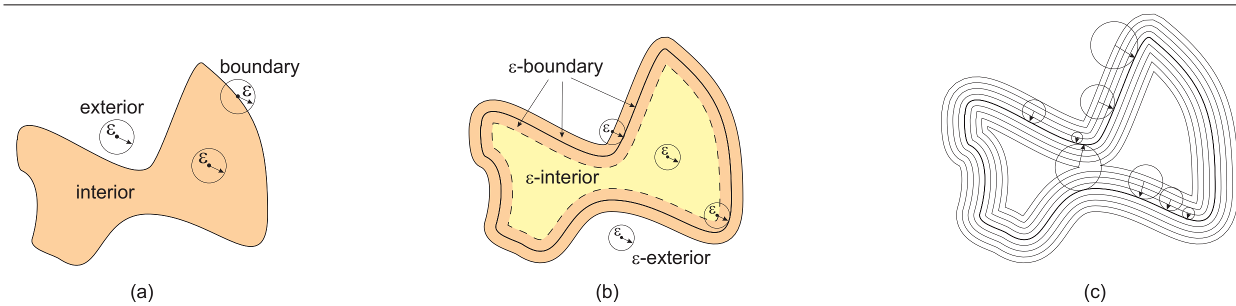

Figure 2: Classical topological operations and the corresponding ε-topological operations. (a) Classical topological operations are defined using infinitesimal ε = 0 neighborhoods. (b) ε-topological operations are defined using finite size neighborhoods. (c) Both ε-interior and ε-exterior decrease, and ε-boundary grows as ball radius ε increases.

Definition 2.1. Given a subset X of a metric space, a point x is said to be in the *ε-closure* of X, denoted $k_\varepsilon(X)$, if for every $r > \varepsilon$, $B(x, r) \cap X \neq \emptyset$, where ε is a non-negative real number.

Definition 2.2. Given a subset X of a metric space, a point x is said to be in the *ε-interior* of X, denoted $i_\varepsilon(X)$, if there exits an open ball about x of radius $r > \varepsilon$ such that $B(x, r) \subseteq X$, where ε is a non-negative real number.

Definition 2.3. Given a subset X of a metric space, a point x is said to be in the *ε-boundary* of X, denoted $\partial_\varepsilon(X)$, if x is in both $k_\varepsilon(X)$ and $k_\varepsilon c(X)$, where $c(X)$ denotes the complement of a set.

The above ε-topological operations are illustrated in Fig. 2. For $\varepsilon = 0$, these operations correspond to the usual classical topological operations, and in this sense, they are generalizations of the corresponding definitions in the general (point-set) topology [17, 19]. But for $\varepsilon > 0$, additional points are added to or subtracted from closure or interior respectively, and the boundary of the set is "thickened" by the ball of radius ε. Notice that operations k_ε and i_ε define sets that are closed and open in the usual metric topology with $\varepsilon = 0$. It is not difficult to show that the ε-topological operations inherit and preserve many of the classical properties, as illustrated by the theorems below. In what follows, $c(X)$ denotes the complement of set X.

Theorem 2.4. *A point x is in the complement of $k_\varepsilon(X)$ if and only if there is an open ball about x of radius $r > \varepsilon$ such that $B(x, r) \cap X = \emptyset$.*

Proof. If $B(x,r) \cap X = \emptyset$ with $r > \varepsilon$, then it is also true that there is a $B(x,r) \cap k_\varepsilon(X) = \emptyset$ with $r > 0$. Therefore, x is not in $k_\varepsilon(X)$. Conversely, if x is in the complement of $k_\varepsilon(X)$, there is a $B(x,r) \cap k_\varepsilon(X) = \emptyset$ with $r > 0$, and therefore $B(x,r) \cap X = \emptyset$ with $r > \varepsilon$. □

Theorem 2.5. $i_\varepsilon(X) = ck_\varepsilon c(X)$.

Proof. If x is a point in $ck_\varepsilon c(X)$, then there is a $B(x,r)$ of $r > \varepsilon$, $B(x,r) \cap c(X) = \emptyset$, or to say $B(x,r) \subseteq X$, so from Definition 2.2, x is a point in $i_\varepsilon(X)$. Conversely, if x is a point in $i_\varepsilon(X)$, then there is a $B(x,r)$ of $r > \varepsilon$, $B(x,r) \subseteq X$, so $B(x,r) \cap c(X) = \emptyset$, so from Theorem 2.4, x is a point in $ck_\varepsilon c(X)$. □

Corollary 2.6. $ci_\varepsilon(X) = k_\varepsilon c(X)$, $k_\varepsilon(X) = ci_\varepsilon c(X)$.

Similarly, it can be shown that many (but not all[1]) classical theorems are preserved under these generalized topological operations [12]. From the above definitions, a point cannot be in both ∂_ε and i_ε. Thus, every set $k_\varepsilon(X)$ is partitioned into its ε-interior and a 'thickened' boundary as $k_\varepsilon(X) = \partial_\varepsilon(X) \cup i_\varepsilon(X)$. It follows immediately that for any set X,

$$i_\varepsilon(X) \subseteq X \subseteq k_\varepsilon(X). \tag{1}$$

The set of points that are *not* in $k_\varepsilon(X)$ is called *ε-exterior* of set X and will be denoted $e_\varepsilon(X)$. It can be also defined directly following Theorem 2.4 using open ball. Since

$$e_\varepsilon(X) \cup \partial_\varepsilon(X) \cup i_\varepsilon(X) = W, \tag{2}$$

we can say that any set $X \subseteq W$ induces a partition of the space W under the ε-topological operations.

Consider what happens to the induced partition of space W under different values of ε (Fig. 2(c)). For a given set X, a larger value of ε will result in a shrunk ε-interior $i_\varepsilon(X)$ and a grown ε-closure $k_\varepsilon(X)$. As a result, the ε-boundary will thicken further as well. We can summarize this concisely by the theorem whose proof follows directly from the above definitions.

Theorem 2.7. *For a given subset X of a metric space, and non-negative numbers ε_1 and ε_2, if $\varepsilon_1 \geq \varepsilon_2$, then*

$$i_{\varepsilon_1}(X) \subseteq i_{\varepsilon_2}(X), e_{\varepsilon_1}(X) \subseteq e_{\varepsilon_2}(X), \partial_{\varepsilon_1}(X) \supseteq \partial_{\varepsilon_2}(X). \tag{3}$$

Theorem 2.7 implies that i_ε and e_ε are monotonically decreasing functions of ε: as ε decreases, points can only be added to the interior $i_\varepsilon(X)$ and exterior $e_\varepsilon(X)$, while they are removed from the shrinking boundary $\partial_\varepsilon(X)$. As ε approaches 0, $e_\varepsilon, i_\varepsilon, \partial_\varepsilon$ approach the classical exact sets of exterior, interior, and boundary respectively.

Many additional properties of ε-topological operations may be verified using straightforward application of set calculus. We summarize the properties that are of particular importance to shape modeling, without giving the detailed proofs. If A and B are arbitrary subsets of Euclidean space, then we have:

- $A \subseteq B \rightarrow k_\varepsilon(A) \subseteq k_\varepsilon(B)$, $\quad A \subseteq B \rightarrow i_\varepsilon(A) \subseteq i_\varepsilon(B)$;
- $k_\varepsilon k_\varepsilon(A) = k_{2\varepsilon}(A)$, $\quad i_\varepsilon i_\varepsilon(A) = i_{2\varepsilon}(A)$;
- $k_\varepsilon(A \cup B) = k_\varepsilon(A) \cup k_\varepsilon(B)$, $\quad i_\varepsilon(A \cap B) = i_\varepsilon(A) \cap i_\varepsilon(B)$;
- $k_\varepsilon(A \cap B) \subseteq k_\varepsilon(A) \cap k_\varepsilon(B)$, $\quad i_\varepsilon(A \cup B) \supseteq i_\varepsilon(A) \cup i_\varepsilon(B)$;
- $k_\varepsilon i_\varepsilon k_\varepsilon i_\varepsilon(A) = k_\varepsilon i_\varepsilon(A)$, $\quad i_\varepsilon k_\varepsilon i_\varepsilon k_\varepsilon(A) = i_\varepsilon k_\varepsilon(A)$.

Other properties of the implied topological spaces follow from [12], but we do not consider them in this paper. Throughout the paper, we rely on the common notions of open and closed sets, as defined in the usual natural topology of Euclidean space E^d. In particular, note that ε-interior $i_\varepsilon X$ and ε-closure $k_\varepsilon X$ of any set X are respectively open and closed in E^d.

[1] Notably, the classical property $k(k(X)) = k(X)$ does not hold for k_ε operation when $\varepsilon > 0$.

2.2 ε-Regular Sets

Intuitively, it should be apparent that a proper definition of regularity must not eliminate but *tolerate* errors and imperfections near the set boundary. The definition should be consistent with the classical notions of regular sets, and in fact the generalization is relatively straightforward. We *postulate* that a *known* set of X should be deemed ε-regular if *an ε-neighborhood of every boundary point $p \in \partial_0(X)$ contains points in the set interior $i_0(X)$ and points in the set exterior $e_0(X)$*. The challenge is to formulate this postulate in terms of the ε-topological operations in a manner that is consistent with the classical notions of regularity. Let us first consider what this means when $\varepsilon = 0$. By definition, $i_0(X)$ is the largest open set contained in X, and $k_0(X)$ is the smallest closed set that contains X, which means that for any set X,

$$i_0(X) \subseteq X \subseteq k_0(X). \tag{4}$$

Neither one of these bounding sets are guaranteed to be homogeneous in dimension: $i_0(X)$ may contain voids and cracks, while $k_0(X)$ may include dangling pieces and isolated points. The homogenization (or regularization) is achieved by a second topological operation that grows the lower and shrinks the upper bounding sets respectively: $k_0 i_0(X)$ is the smallest closed regular set containing $i_0(X)$, and $i_0 k_0(X)$ is the largest open regular set contained in $k_0(X)$. The regularization effectively reverses the set relationship (4) to

$$i_0 k_0(X) \subseteq X \subseteq k_0 i_0(X). \tag{5}$$

It should be clear that any open or closed regular set X satisfies the inequality (5) whose essence is to combine the two classical definitions into a single definition of an 0-regular set. But other sets satisfy the above inequality as well, because it allows for imperfections (missing portions or isolated points) in the boundary $\partial_0(X)$. For example, a unit sphere with half of its boundary missing would satisfy (5) and be considered 0-regular.

However, the two homogeneous bounds (5) are so tight that they are not realistic, because no set X satisfying (5) may contain even minor imperfections in its interior or exterior, such as those shown in Fig. 1. To say that imperfections are tolerated in the interior of the set X *within distance* ε from the boundary amounts to a statement that X contains $i_\varepsilon k_0(X)$; similarly, when $k_\varepsilon i_0(X)$ contains the set X, exterior imperfections within ε from the boundary are tolerated. This motivates our first attempt at definition of ε-regularity for a *known* set X.

Definition 2.8. A subset X of a metric space is *ε-regular* if, for a given non-negative real number ε,

$$i_\varepsilon k_0(X) \subseteq X \subseteq k_\varepsilon i_0(X). \tag{6}$$

Informally, the above definition assumes that we can determine the closure and interior of X exactly, but we allow errors within distance ε near the boundary. The concept is illustrated in Fig. 3 showing that imperfections within ε of the boundary in Fig. 1 are "covered" by either growing $i_0(X)$ by k_ε as in Fig. 3(a), or by shrinking $k_0(X)$ by i_ε as in Fig. 3(b). Thus, in contrast to the classical definitions, under the proposed notion of ε-regularity, these representations of rectangles are considered to represent ε-regular sets.

By Definition 2.8, ε-regularity of a set depends on the size of chosen ε. Classical regular sets (open and closed) clearly satisfy the definition in the special case when $\varepsilon = 0$, with one side of (6) becoming an equality. Inequality is crucial in the general case, because it corresponds to the notion of tolerant modeling near the boundary of the set. By definition, increasing the value of ε corresponds to shrinking $i_\varepsilon k_0(X)$ and expanding $k_\varepsilon i_0(X)$. Thus, if X is ε-regular, it is also ε_1-regular for any $\varepsilon_1 \geq \varepsilon$. We observe that every bounded set (or rather every set containing bounded 'imperfections') with non-empty interior is ε-regular for some sufficiently large value of ε. Normally, we are interested in the smallest value of ε for which X is ε-regular.

2.3 ε-Regular Intervals

We now consider more realistic situations where we may not be able to compute the interior $i_0(X)$ and closure $k_0(X)$ of a set X exactly, but only some bounding sets can be determined – either as input or as a result of another approximate computation. In what follows, we will refer to the bounding set X_- as *inner*, and X_+ as *outer*, assuming the inner is open and the outer is closed. An example of inner and outer bounding sets is shown in Fig. 3(c). In this sense, the interior $i_0(X)$ and the closure $k_0(X)$ are the tightest bounds computable for any given set X. The inner and outer sets form a *set interval*, $[X_-, X_+]$ — *the class of sets* $\{X\}$ such that $X_- \subseteq X \subseteq X_+$. Then Definition 2.8 generalizes in a straightforward fashion by replacing $k_0(X)$ and $i_0(X)$ respectively with arbitrary outer X_+ and inner X_-:

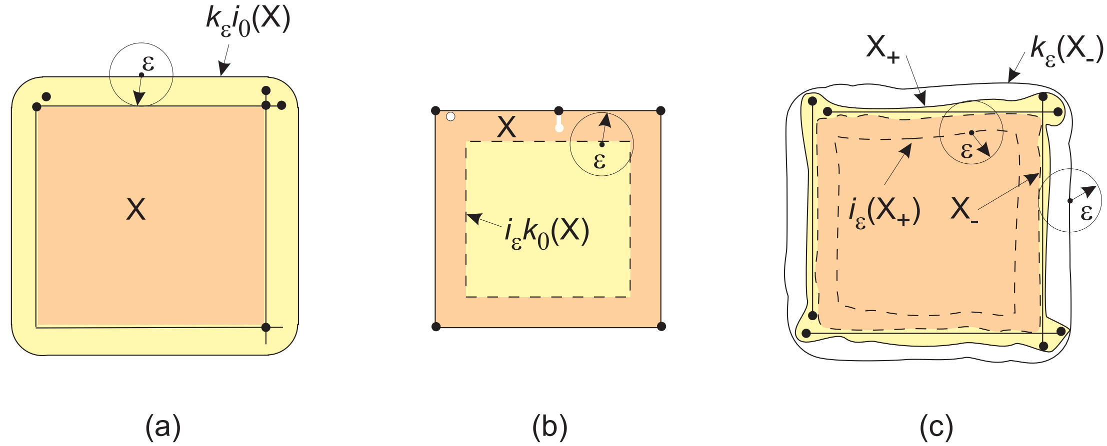

Figure 3: ε-regular sets and intervals tolerate imperfections of size less than ε near the set boundary: (a) growing $i_0(X)$ to cover dangling edges and isolated points; (b) shrinking $k_0(X)$ to cover small cracks and voids; (c) an interval $[X_-, X_+]$ covers the imperfection by $i_\varepsilon(X_+)$ and $k_\varepsilon(X_-)$.

Definition 2.9. A set interval $[X_-, X_+]$ is ε-*regular* if, for a given non-negative real number ε,

$$i_\varepsilon(X_+) \subseteq X_- \subseteq X_+ \subseteq k_\varepsilon(X_-). \tag{7}$$

Technically, the test for ε-regularity of an interval depends on two separate conditions illustrated in Fig. 4: $i_\varepsilon(X_+) \subseteq X_-$ requires that when outer X_+ is shrunk by a ball of size ε, it fits inside the inner; similarly $X_+ \subseteq k_\varepsilon(X_-)$ requires that inner X_- grown by a ball of size ε contains outer X_+. For example, the rectangle in Fig. 3(c) is not regular in the classical sense, but is ε-regular as an interval $[X_-, X_+]$. Once again, if the interval is ε-regular for any particular value of ε, then it must also be ε-regular for any greater value of ε, but not necessarily for the smaller.

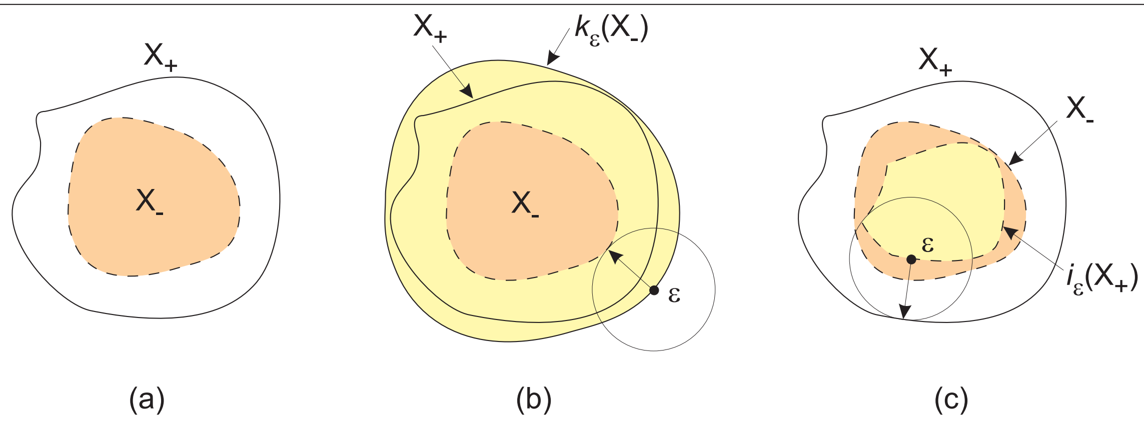

Figure 4: A set interval becomes ε-regular if the ε is big enough: (a) set interval $[X_-, X_+]$; (b) $X_+ \subseteq k_\varepsilon(X_-)$; (c) $i_\varepsilon(X_+) \subseteq X_-$.

Definition 2.9 for a set interval is written in the same form as Definition 2.8 for a set instance, in order to emphasize their common structure. In fact, it is easy to show that Definition 2.8 is a special case of Definition 2.9, by recalling that any set X is contained in the interval $[i_0(X), k_0(X)]$.

Theorem 2.10. *A set X is ε-regular iff the interval $[i_0(X), k_0(X)]$ is ε-regular.*

Proof. From Definition 2.8, we have $k_\varepsilon i_0(X) \supseteq k_0(X)$, and $i_0(X) \supseteq i_\varepsilon k_0(X)$. Thus, let $X_- = i_0(X)$ and $X_+ = k_0(X)$. For the interval $[X_-, X_+]$, we have $k_\varepsilon X_- \supseteq X_+$, and $X_- \supseteq i_\varepsilon X_+$, which is an ε-regular interval. Conversely, if $[i_0(X), k_0(X)]$ is ε-regular, then from Definition 2.9, $k_\varepsilon i_0(X) \supseteq k_0(X)$, $i_0(X) \supseteq i_\varepsilon k_0(X)$. Thus, X is ε-regular. □

In other words, we really need only Definition 2.9 of ε-regular interval, because it subsumes Definition 2.8 of an ε-regular set. Henceforth, it should be understood that the term 'ε-regular interval' also applies to ε-regular set instances. Furthermore, it is easy to see that every set instance X in an ε-regular interval $[X_-, X_+]$ is also ε-regular. This is reasonable and should be expected, since every such set interval represents an equivalence class of sets that are not distinguishable beyond the inner and outer bounds of the interval. In fact, if we define a *subinterval* $[Y_-, Y_+]$ of interval $[X_-, X_+]$ as *the class of sets* $\{Y\}$ such that $Y_- \subseteq Y \subseteq Y_+$, with the inner $Y_- \supseteq X_-$ and the outer $Y_+ \subseteq X_+$, we can make an even stronger claim:

Theorem 2.11. *Any subinterval $[Y_-, Y_+]$ of an ε-regular interval $[X_-, X_+]$ is also ε-regular.*

Proof. By definition, X_-, Y_- are open, and X_+, Y_+ are closed sets, with $Y_- \supseteq X_-$ and $Y_+ \subseteq X_+$. Since $k_\varepsilon(X_-) \supseteq X_+$, then $k_\varepsilon(Y_-) \supseteq k_\varepsilon(X_-) \supseteq X_+ \supseteq Y_+$; similarly, since $X_- \supseteq i_\varepsilon(X_+)$, then $Y_- \supseteq X_- \supseteq i_\varepsilon(X_+) \supseteq i_\varepsilon(Y_+)$. Thus, $[Y_-, Y_+]$ is ε-regular. □

This result is of paramount *practical* significance, because it allows to verify regularity of an interval $[Y_-, Y_+]$ even when the interval itself is not computable by testing a larger containing interval $[X_-, X_+]$ that is computable. In particular, Fig. 5 shows that any set X contained in an ε-regular interval must be ε-regular. This statement is conservative in a sense that X may be ε-regular with even a smaller value of ε.

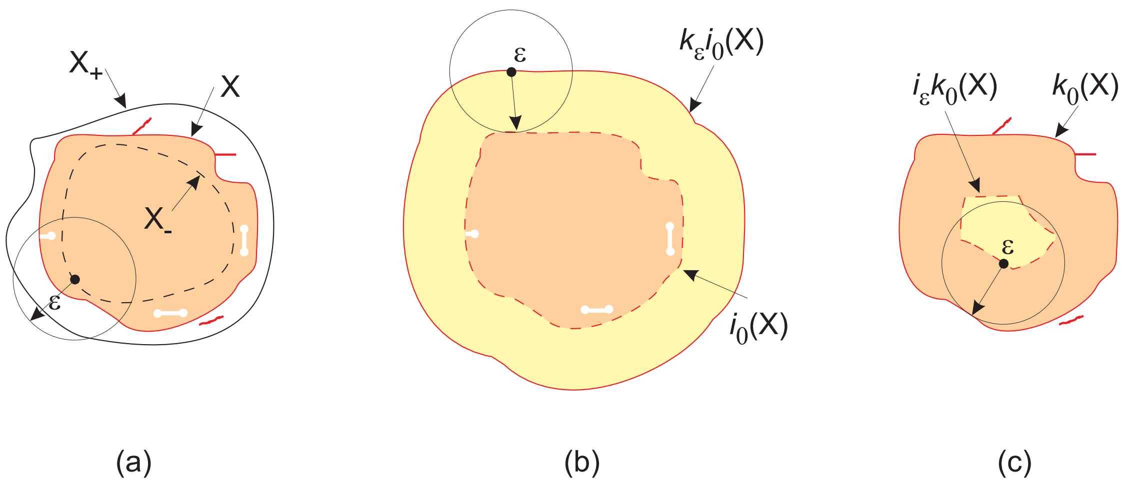

Figure 5: Any set contained in an ε-regular interval is ε-regular with the same or smaller value of ε: (a) set instance X with dangling pieces and inner cracks, contained in an ε-regular set interval; (b) $X \subseteq k_\varepsilon i_0(X)$; (c) $i_\varepsilon k_0(X) \subseteq X$.

3 Properties of ε-Regular Intervals

3.1 Connections with Morphological Operations

There appears to be every reason to believe that morphological operations, widely used in image processing, are closely related to the proposed model of regularity, because they effectively filter out small 'noisy' features of finite size from images [28, 14]. It is intuitive and has been observed by others that these operations are related to regularization [27, 28, 4]. Below we demonstrate the subtle differences between ε-topological operations and morphological operations, and explain why morphological operations are not suitable for modeling of ε-regular shapes.

For a comprehensive introduction to mathematical morphology, the reader is referred to the classic monograph by Serra [28]. Here, we only briefly summarize the key concepts. The *dilation* of a set A by a structuring element B is defined as

$$D_B(A) = A \oplus \check{B} = \{x : B_x \cap X \neq \emptyset\}, \tag{8}$$

where $\oplus$ denotes the Minkowski addition and $\check{B} = \{-b : b \in B\}$ is the transpose of set B. Similarly, the *erosion* of a set A by a structuring element B is

$$E_B(A) = A \ominus \check{B} = \{x : B_x \subseteq X\}, \tag{9}$$

where $\ominus$ denotes the Minkowski subtraction. By combining erosion and dilation, Matheron [18] proposed the notions of morphological *opening* A_B and *closing* A^B as follows:

$$A_B = (A \ominus \check{B}) \oplus B, \qquad A^B = (A \oplus \check{B}) \ominus B. \tag{10}$$

Informally, dilation and erosion are respectively 'growing' and 'shrinking' operations, while opening and closing are obtained by sequential application of dilation and erosion to a given set. Furthermore, the following properties are known to hold for opening and closing of any sets A and B:

- increasing, $A \subseteq A'$ implies $A_B \subseteq A'_B$ and $A^B \subseteq A'^B$;
- idempotent, $(A_B)_B = A_B$ and $(A^B)^B = A^B$;
- duality, $(C(A))_B = C(A^B)$ and $(C(A))^B = C(A_B)$;
- extensive (closing) and anti-extensive (opening), $A_B \subseteq A \subseteq A^B$.

The above operations have found numerous applications in geometric and solid modeling, notably for offsetting, blending and filleting operations [27], reconstruction [4], and shape simplification [33].

It may appear that the proposed operations of ε-closure and ε-interior are special cases of the morphological dilation and erosion respectively, when we restrict the structuring element B to be a ball of radius ε. Just like the morphological operations, ε-topological operations grow or shrink a set X by the ball of size ε. However, this intuitive correspondence does not hold under more careful analysis, because the Minkowski operations used in the definitions of dilation and erosion do not capture the expected topological properties. In general, the dilation or erosion of a set by closed ball may be neither topologically closed nor open. When we restrict structuring element B to be an open ball, dilation always results in an open set and erosion always gives a closed set. Figures 6(b) and 6(c) illustrate respectively the dilation and erosion of a closed set A by an open ball B. Clearly, the dilation (erosion) of set A does not correspond to ε-closure (ε-interior) of A because the latter must be closed (open). Furthermore, the dilation (erosion) of an open (closed) set by a closed ball is identical to its dilation by an open ball of the same radius.

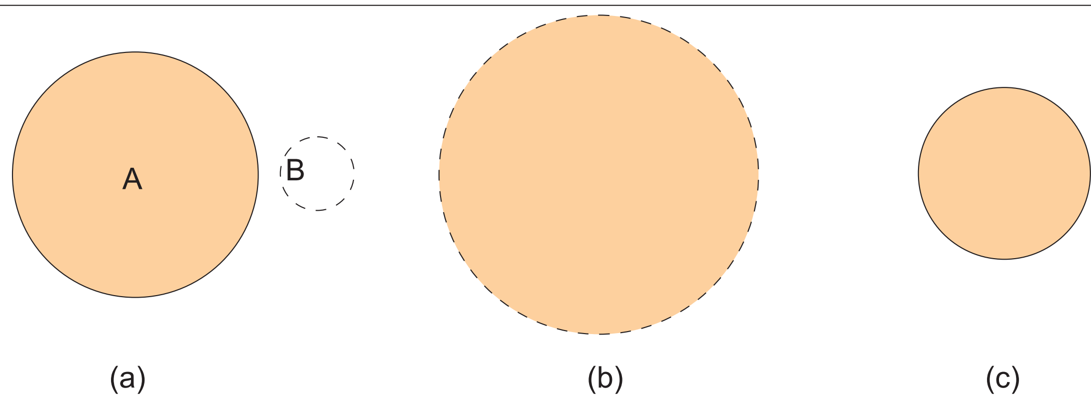

Figure 6: Morphological dilation and erosion of a closed set A by open ball B: (a) closed A and open B; (b) dilation of A by B is open; (c) erosion of A by B is closed.

The relationship between ε-topological operations and morphological operations is established by the following equations:

$$k_\varepsilon(X) = k_0(D_B(X)), \qquad i_\varepsilon(X) = i_0(E_B(X)). \tag{11}$$

These equations are easy to understand intuitively: ε-closure is the 0-closure of the dilation of X by ball B, and ε-interior is the 0-interior of the erosion of X by ball B. These rather technical differences are also the main reasons why morphological operations cannot be used to formulate the ε-regularity concept proposed in this paper. For example, we have already shown earlier that the sequences of ε-topological operations $k_\varepsilon i_\varepsilon$ and $i_\varepsilon k_\varepsilon$ satisfy the increasing, idempotent, and duality properties of the opening and closing operations respectively. However, it is generally not true that $k_\varepsilon i_\varepsilon(X) \subseteq X \subseteq i_\varepsilon k_\varepsilon(X)$. Consider the rectangle X in Fig. 7(a). The rectangle itself could be open, closed, or neither open nor closed, depending on whether or not the rectangle includes its boundary. By definition, $k_\varepsilon i_\varepsilon(X)$ is the closed set with rounded corners, while $i_\varepsilon k_\varepsilon(X)$ is the interior of the rectangle (an open set). Therefore, it follows that *no* rectangles satisfy the inequality $k_\varepsilon i_\varepsilon(X) \subseteq X \subseteq i_\varepsilon k_\varepsilon(X)$. Thus, it should be clear that the composite operation $k_\varepsilon i_\varepsilon$ is not anti-extensive, and $i_\varepsilon k_\varepsilon$ is not extensive. In contrast, Fig. 7(b) shows that the corresponding opening and closing of X by an open ball B guarantee that $X_B \subseteq X \subseteq X^B$, since X_B is always open and X^B is always closed for any X.

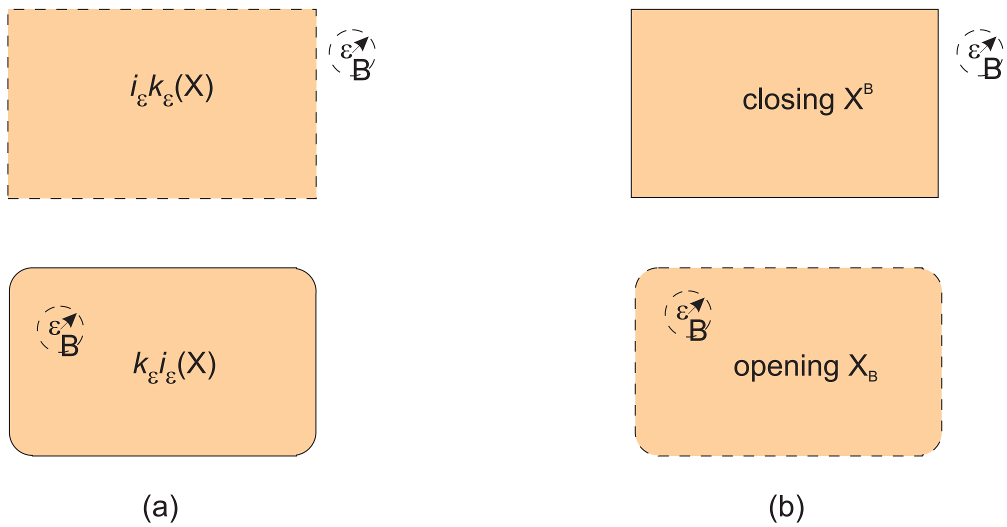

Figure 7: (a) Topological $k_\varepsilon i_\varepsilon$ and $i_\varepsilon k_\varepsilon$ operations violate $k_\varepsilon i_\varepsilon(X) \subseteq X \subseteq i_\varepsilon k_\varepsilon(X)$. (b) Morphological opening and closing preserve $X_B \subseteq X \subseteq X^B$.

There have been several proposals to use morphological operations to generalize the notion of regularity, based on the observations that opening and closing filter out small features of size ε. For example, the authors in [4, 33] proposed several types of regularity for X depending on whether $X = X_B$ and/or $X = X^B$. Such conditions may hold in special situations arising in shape reconstruction and simplification, but they are not useful for our purposes. If B is a closed set, then it is possible that $X = X_B = X^B$ if X belongs to a special subclass of classical regular sets. However, if B is an open set, then the opening X_B is an open set, while the morphological closing X^B is a closed set; therefore there are *no* non-empty sets satisfying $X = X_B = X^B$. In this case, regularity could be defined as $X = X_B$ or $X = X^B$, but again these conditions hold only for some subclass of open or closed regular sets X. Figure 8 shows that, in general, neither opening nor closing are guaranteed to be regular sets in general. To summarize, in all cases, the morphological regularity *restricts* the classical notions of closed and open regularity respectively, by eliminating regular sets with features of small but finite size ε.

In contrast, our goal is not to eliminate small features and imperfections, but to tolerate them. Because the relationship $X_B \subseteq X \subseteq X^B$ holds for any set X, the opening X_B and the closing X^B naturally suggest themselves as the candidate sets tolerating bounds on the set X. However, it should be clear from Fig. 8 that the opening and closing themselves certainly do not define an ε-regular interval. The essential condition of ε-regularity comes not from the morphological properties of X but from the additional containment requirements given in Definition 2.9. As we show next, this definition also imposes Hausdorff distance properties that are not implied by the morphological operations.

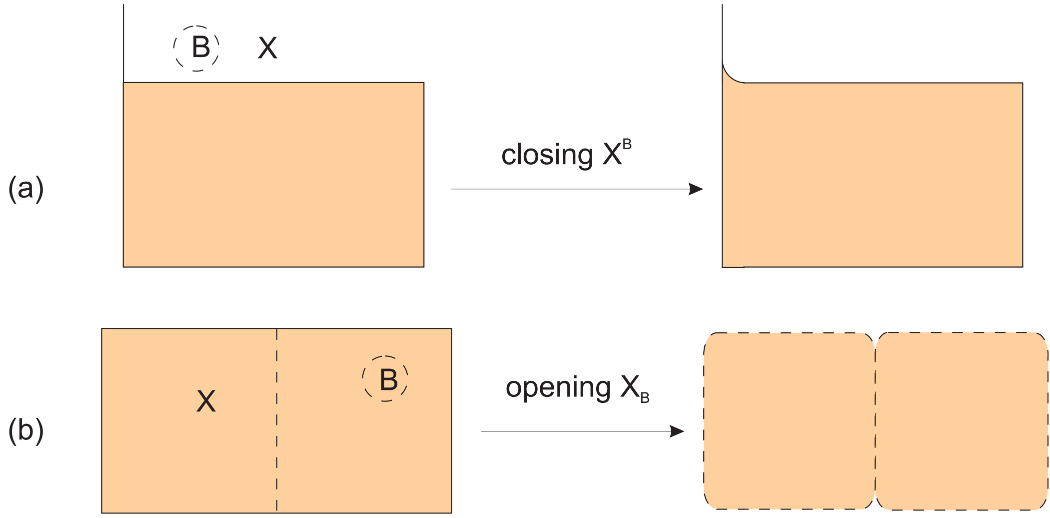

Figure 8: Morphological opening and closing are not topologicaly open and closed regular sets: (a) closing of a rectangle with dangling edge is closed set but not closed regular; (b) opening of a rectangle with inner crack is open set but not open regular.

3.2 The Hausdorff Distance Property

The Hausdorff distance between two sets is the standard tool in shape comparison, matching, and reconstruction. Let X and Y be two non-empty closed bounded subsets of a metric space (W, d), The Hausdorff distance is given by

$$d_H(X, Y) = \max\{d_h(X, Y), d_h(Y, X)\}, \tag{12}$$

where $d_h(X, Y) = \sup_{x \in X} \inf_{y \in Y} d(x, y)$ is the directed Hausdorff distance from X to Y. Here, we use the Hausdorff distance on the closed subsets of W [22]. (If sets are not required to be bounded, the Hausdorff distance may take infinite value.)

It is well known that there are many circumstances when Hausdorff distance is not effective without additional assumptions, because qualitatively dissimilar sets may be within small distance of each other [9, 31]. As a partial remedy, Boyer and Stewart [9] proposed the d_w metric as the maximum value of the Hausdorff distance between sets and the Hausdorff distance between the 0-boundaries of sets. In practice, both types of measures have been used, for example see [2]. We now show that all sets within an ε-regular interval are within ε Hausdorff distance and, furthermore, the maximum distance between their boundaries is also at most ε. In other words, ε-regularity imposes a strong similarity measure on all sets in the interval and subsumes the usual Hausdorff measures proposed by others.

By definition, an ε-regular set interval $[X_-, X_+]$ is bounded by two sets X_- and X_+, satisfying the ε-regularity condition (2.9), with $X_- \subseteq X_+$. It suffices to bound the Hausdorff distance between X_- and X_+ and their boundaries. Since X_- is open, we consider the Hausdorff distance between the closed sets $k_0(X_-)$ and X_+.

Theorem 3.1. *For an ε-regular interval $[X_-, X_+]$, the Hausdorff distance $d_H(k_0(X_-), X_+) \leq \varepsilon$.*

Proof. We first prove $d_h(k_0(X_-), X_+) \leq \varepsilon$. By definition, $k_0(X_-)$ is the smallest closed set containing the open set X_-. Since X_+ is also a closed set containing X_-, $k_0(X_-) \subseteq X_+$, and therefore the directed Hausdorff distance $d_h(k_0(X_-), X_+) = 0$. By definition, $d_h(k_\varepsilon(X_-), k_0(X_-)) \leq \varepsilon$. Since $X_+ \subseteq k_\varepsilon(X_-)$, the directed Hausdorff distance $d_h(X_+, k_0(X_-)) \leq \varepsilon$. The theorem follows. □

Corollary 3.2. *For an ε-regular interval $[X_-, X_+]$, the Hausdorff distance $d_H(c(X_-), k_0c(X_+)) \leq \varepsilon$.*

Proof. From the definition of ε-regular interval, if $[X_-, X_+]$ is ε-regular, then so is $[c(X_+), c(X_-)]$. The corollary follows. □

In other words, ε-regularity implies the Hausdorff distance of ε not only between the bounding sets X_- and X_+, but also between their complements. Figure 9(a) demonstrates that this is a rather strong condition that does not hold for an arbitrary set interval. In fact, Theorem 3.1 and Corollary 3.2 suggest that ε-regularity can be characterized in terms of Hausdorff distances. Specifically, it can be shown that two sets $X_- \subseteq X_+$ form an ε-regular interval $[X_-, X_+]$ only if the Hausdorff distance between the two sets and their complements is bounded by ε. Furthermore, it is now easy to prove that the Hausdorff distance between the boundaries of the bounding sets is also at most ε.

Theorem 3.3. *For an ε-regular interval $[X_-, X_+]$, the Hausdorff distance $d_H(\partial_0(X_-), \partial_0(X_+)) \leq \varepsilon$.*

Proof. We first prove $d_h(\partial_0(X_+), \partial_0(X_-)) \leq \varepsilon$. Instead of computing the directed Hausdorff distance from $\partial_0(X_+)$ to $\partial_0(X_-)$ directly, it is easier to show that the directed distance from the larger set $\partial_\varepsilon(X_-)$ to $\partial_0(X_-)$ is bounded by ε. This will prove that $d_h(\partial_0(X_+), \partial_0(X_-)) \leq \varepsilon$, based on the definition of the directed Hausdorff distance.

Let us show that $\partial_\varepsilon(X_-)$ indeed contains $\partial_0(X_+)$. Since X_+ is closed, $\partial_0(X_+) \subseteq X_+$, and since the interval is ε-regular, $X_+ \subseteq k_\varepsilon(X_-)$. Therefore $\partial_0(X_+) \subseteq k_\varepsilon(X_-)$. Since no point of $\partial_0(X_+)$ can be inside X_-, it follows that $\partial_0(X_+) \subseteq c(X_-)$. Also, by definition of k_ε, $c(X_-) \subseteq k_\varepsilon c(X_-)$. Thus, $\partial_0(X_+) \subseteq k_\varepsilon c(X_-)$. Therefore it is clear that $\partial_0(X_+) \subseteq \partial_\varepsilon(X_-)$. But $d_h(\partial_\varepsilon(X_-), \partial_0(X_-)) \leq \varepsilon$, by definition of ∂_ε.

The proof that $d_h(\partial_0(X_-), \partial_0(X_+)) \leq \varepsilon$ is very similar. Since $\partial_0(X_-) \subseteq \partial_\varepsilon(X_+)$, and $d_h(\partial_\varepsilon(X_+), \partial_0(X_+)) \leq \varepsilon$, it must be that $d_h(\partial_0(X_-), \partial_0(X_+)) \leq \varepsilon$.

□

The above results show that ε-regularity imposes a stronger distance condition than is implied by both the Hausdorff distance and the d_w metric. This relationship is not reciprocal. Figure 9(b) shows a simple example where $[X_-, X_+]$ is ε-regular for ε values that must be larger than both the Hausdorff distance between sets X_-, X_+ and the Hausdorff distance between their boundaries. Intuitively, ε-regularity provides a stronger measure of shape similarity because it is symmetric with respect to the interval $[X_-, X_+]$ and the complementary set interval, as indicated by the Corollary 3.2 above.

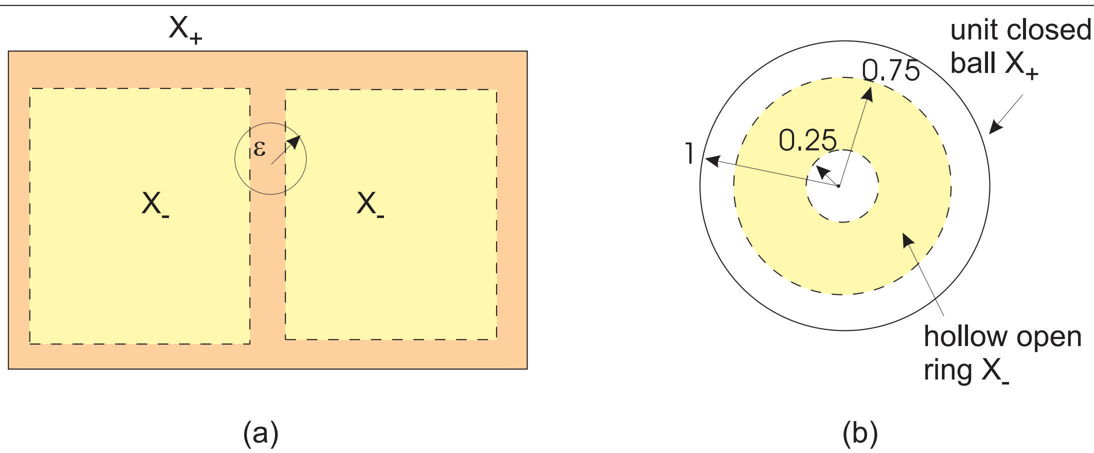

Figure 9: An ε-regular set provides stronger distance condition than the Hausdorff distance between sets and the Hausdorff distance between boundaries of sets. (a) $d_H(k_0(X_-), X_+)$ is within ε, while $d_H(c(X_-), k_0c(X_+))$ is bigger than ε. (b) $d_H(X_-, X_+)$ is 0.25; $d_H(\partial_0(X_-), \partial_0(X_+))$ is 0.75; the interval $[X_-, X_+]$ is ε-regular for $\varepsilon \geq 1$.

4 Significance and Applications

This paper discusses a generalization of the classical regularity models, based on the observation that topological properties of sets may be represented and/or computed only within some finite precision[23]. Importantly, the formulated

ε-regularity subsumes the classical regularity formulations as special (but not very realistic) cases of $\varepsilon = 0$. In this sense, the ε-regularity model captures more realistically the practices and the recognized limitations of geometric and solid modeling.

We showed the connections and distinctions between the ε-topological operations and the common morphological operations. From topological point of view, ε-closure k_ε always generates closed sets and ε-interior i_ε always generates open sets, while morphological dilation and erosion do not have these properties. From algebraic point of view, the ε-topological operations and the morphological operations share many (but not all) common properties that are useful for shape modeling, simplification, and reconstruction. In particular, ε-topological operations allow quantification of shape irregularities, while morphological regularity applies only to classical closed/open regular sets. Last but not least, we characterized an ε-regular interval in terms of its Hausdorff distance properties and showed that ε-regularity implies a stronger measure of shape similarity than the previously proposed Hausdorff-distance based measures.

We conclude by briefly explaining how the concept of ε-regularity may be applied to a number of outstanding problems, where geometric models cannot be *represented* and/or *computed* exactly. The key premise is that most problems involving uncertainly or approximation in geometric data or computation may benefit from reformulation in terms of ε-regular *intervals*. In other words, if representations of classical models and/or the evaluation algorithms for representations are not exact, then our modeling space is a set of ε-regular intervals, with each interval representing all of its subsets and sub-intervals, including the classical open and closed regular sets as special cases. For example, suppose we have a representation of some set X, but its exact interior $i_0(X)$ and closure $k_0(X)$ cannot be computed exactly, either because of imprecision in data or for the lack of exact arithmetic evaluation procedure. Theorem 2.11 implies that, for practical purposes, X may be treated as ε-regular set if it is represented by any ε-regular interval $[X_-, X_+]$ containing X. In practice, we often encounter the following cases and their combinations, listed in the order of increasingly tight intervals:

1. $X_- = i_\delta(X)$ and $X_+ = k_\delta(X)$, where $\delta > 0$ is a real number representing default precision of a point membership test [23].

2. $X_- = i_\delta(X) \cup \{p\}$, $p \in (i_0(X) \cap \partial_\delta(X))$; $\quad X_+ = k_\delta(X) \setminus \{q\}$, $q \in (e_0(X) \cap \partial_\delta(X)$. In other words, the default interval $[i_\delta(X), k_\delta(X)]$ can be tightened by adding to X_- or removing from X_+ points from $\partial_\delta(X)$ with known membership classification.

3. $X_- = i_{\delta'}(X)$ and $X_+ = k_{\delta'}(X)$, where $0 < \delta' < \delta$ is a function of spatial location, representing variable resolution of computation.

4. X_- is the opening of X by open δ ball, $X_- = X_B$; X_+ is the closing of X by open δ ball, $X_+ = X^B$.

In the above list, the first case of $[i_\delta(X), k_\delta(X)]$ demands the largest value of ε, with $\varepsilon \geq \delta$. However, if we are able to establish that $[i_\delta(X), k_\delta(X)]$ is ε-regular, then so are the rest of the subintervals, including X itself. Tighter intervals correspond to more precise computations, and possibly smaller value of ε. Ideally, we would like to know the smallest value of ε for which $[X_-, X_+]$ is ε-regular. Notice that we included morphological operations of opening and closing by an open δ-ball as means for inducing $[X_-, X_+]$; but δ determines how closely X is represented by $[X_-, X_+]$ and does not directly determine ε.

Various combinations of the above interval-inducing techniques are useful for problems in shape model reconstruction, simplification, comparison, validation, repair, and translation. In all cases, the notion of ε-regularity allows to tolerate and ignore small features and errors that are contained in $X_+ \setminus X_-$. For instance, a major challenge in geometric data translation is to develop a suitable formal model which can embrace and tolerate the inevitable changes in data and algorithm precisions. In [23], we proposed the notion of ε-solid model, defined as a ε-regular set interval $[X_-, X_+]$ with non-empty X_- and bounded X_+. The data translation problem then reduces to ascertaining that the translated model is ε'-solid in the receiving system, given that the original model is an ε-solid in the sending system. See [23] for systematic treatment of this question and classification of possible problems, based on possible changes to data accuracy and algorithm precision.

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