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Topological Formulation of Tolerant Solid Modeling

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Abstract

Classical theory of solid modeling relies on the notion of regular sets and presupposes exactness in both geometric data and algorithms. In contrast, modeling, exchange and translation of geometric models in engineering applications usually involve data approximations and algorithms with different numerical precisions. We argue that an appropriate formulation of these geometric modeling problems require finite size neighborhoods, leading to the notion of ε-topological operations. These operations are then used to formulate the definitions of ε-regularity and ε-solid that extend and subsume the corresponding classical concepts as exact special cases. Furthermore, the proposed theory suggests how the classical solid modeling paradigm should be extended in order to deal with the outstanding problems in geometric robustness, validation, and data translation. In particular, it explains why the current methods for validating boundary representations are not always sufficient and demonstrates that widely adapted geometric repairs are often unnecessary for maintaining solidity in the presence of numerical inaccuracies.

Keywords: Solid Modeling, ε-Topological Operations, ε-Regularity, ε-Solidity, Geometric Robustness, Geometric Data Translation, Tolerant Modeling

1 Introduction

1.1 Motivation

The theoretical foundations of modern solid modeling systems were laid in the 1970s at the University of Rochester's Production Automation Project (PAP) [26, 27, 29]. These include the now widely accepted notion that a suitable set-theoretic model for a solid object is an r-set, defined as bounded, closed, regular, and semi-analytic subset of E^d. The notion of (point-set topological) regularity is central to this model. Formally, a set X is called closed regular [18], if $X = ki(X)$, where k and i denote the topological closure and interior of a set X respectively. Informally, this implies that solid's boundary is an infinitesimally thin ("nowhere dense") layer that has no gaps or 'dangling pieces'. All modern solid modeling representations and algorithms are interpreted in terms of this mathematical model. For example, the notions of uniqueness, validity and consistency of geometric representations are interpreted in terms of the existence and/or exact equality of represented r-sets [27].

The above model is usually (and rightfully) credited with the success of solid modeling, but it also introduced fundamental limitations that have remained and amplified over the years. In particular, the definition of regularity requires the ability to represent and compute the topological properties (interior, closure, boundary) exactly. This, in turn, restricts the applicability of this classical theory only to exact geometric data and requires the ability to compute with infinitesimally small neighborhoods. In contrast, most of engineering data involves some uncertainly, most representations are only approximate, and the vast majority of geometric algorithms can be implemented only with finite precision. Thus, in practice, individual geometric modeling systems use small numerical constants ε to construct "tolerant models" and to achieve "robustness" in an ad-hoc fashion. Furthermore, such capabilities in a single system do not provide any guarantee for the exchange and translation of geometric models between different

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modeling systems because of the "interoperability" problem: a "valid" geometric model from one modeling system often cannot be processed in another modeling system due to the mismatch in data accuracy and algorithm precision.

The classical solid modeling theory does not account for these facts, and in particular, the notion of algorithm precision is completely missing. A fundamental research issue in these problems is to address the question: *what is an appropriate mathematical model of solidity when data and/or algorithms are imprecise?* This paper fills this gap in solid modeling theory by proposing the notions of ε-topological operations, ε-regularity, and ε-solidity. Importantly, the classical model remains an important special, albeit usually unrealistic, case of the proposed theory. This paper is based on an earlier technical report by the authors [23]; applications of the proposed theory to the problem of interoperability and data translation are described in [25], and additional properties of ε-regular sets have been studied in [24].

1.2 Related Work

Dealing with data accuracy and algorithm precisions is a long standing problem in geometric robustness research. Typically the errors in a geometric algorithm arise from several sources: the accuracy of input data; the use of finite precision arithmetics to replace real number arithmetic in numerical computations; and inducing logical conclusions and/or combinatorial structures from numerical results. For example, Boolean set operations on boundary representations need to compute the intersections of surfaces (curves) using numerical procedures, and these approximate numerical results are then used to induce the combinatorial data structure of the final boundary representation (usually using a sequence of point membership classification (PMC) tests) [35]. In such an algorithm, the numerical computations are controlled by logic decisions (typically in a form of *if/then* branches) that are invoked depending on the values of ε. As pointed out in [14], the main difficulties of geometric robustness problems lie in the conversion from numerical data to symbolic data: "numerical accuracies introduced either in the initial data or in the finite precision arithmetic that is used, may result in a set of logical decisions that are inconsistent".

Many approaches have been proposed to address data accuracy and algorithm precision problems to achieve robustness in solid modeling systems, notably exact computation approaches [12, 17], and approaches using numerical tolerances [11, 15, 16, 33]. Despite considerable progress, the unsolved fundamental questions remain for each of the current approaches. In particular, the exact computation approaches rely on the assumption that *input data is exact* [40]. This assumption may be acceptable for simple computational geometry problems, such as convex hull and Delaunay triangulation computations where the robustness/correctness is defined in terms of the algorithm precisions used in predicates; but in most engineering problems, the data is intrinsically imprecise since it is obtained through physical experimentations or numerical computations. The correctness of all proposed approaches is usually argued based on an assumption or a mathematical proof of the *existence of an ideal correct result*, which appears to be a difficult problem by itself. For instance, Hopcroft and Kahn [14] gave the existence proof for the intersection of a convex polyhedron and a half-space, while proofs for other types problems are not known. Furthermore, the mere existence of this ideal outcome does not imply that the result is computationally tractable, which is yet another problem.

The issues of data accuracy and algorithm precision also dominate the area of geometric data translation between different systems, that make different assumptions and use widely varying numerical constants in their representations and algorithms. Various geometric healing approaches have been proposed to correct "invalid" boundary representations to "valid" boundary representations [4,5,7,9,21,36,37]. We use quotes, because in healing the notion of validity appears to be implied by specific computations, procedures, modeling tasks and even systems; arguing that the healed representations correspond to some specific r-set models appears to be difficult if not impossible.

We propose to approach the problem of dealing with data and algorithm precisions from another perspective. We accept uncertainty of data and limited accuracy of algorithms as given. Then, it is reasonable to assume that a given geometric representation corresponds to a *class* of geometric objects. We no longer need to prove the existence of any particular idealized r-set, and instead focus on the properties of the class implied by the data and algorithms.

1.3 Outline

In order to deal with inexactness of data and algorithms, in Section 2, we generalize the classical set topological notions of *interior* i, *closure* k, and *boundary* ∂ in terms of finite size neighborhoods. The size of a neighborhood ε relates to the precision of data and algorithms; when precision is limited, the size must be finite. The resulting generalized

ε-topological operations i_ε, k_ε, and ∂_ε satisfy many of the classical topological properties, and provide the basis for the formal definitions of ε-regular sets and ε-solids that explicitly recognize the role of imprecision. The classical r-set solid model proposed by Requicha [27] is shown to be a special case where $\varepsilon = 0$ corresponds to arbitrarily small neighborhoods.

Practical consequences of the proposed theory should become clear in Section 3. The precision in Point Membership Classification (PMC) is formalized under the topological operations of $i_\delta(X)$, $e_\delta(X)$ and $\partial_\delta(X)$. The relationship between data accuracy λ, algorithm precision δ and the solidity ε is established, and a paradigm of ε-solid modeling is illustrated. We show that, under the proposed model, popular solid validity-checking procedures may not be adequate and common geometric healing techniques may not be necessary, when either solid representation or evaluation algorithms are limited in precision.

Open issues and promising extensions of the proposed approach are considered in Section 4. We also discuss broader implications of the proposed formal model of ε-solidity and its relation to classical solid modeling, geometric robustness, and other theories.

2 Solidity with Finite Neighborhoods

In this section, we propose a theory that captures the notions of inexact data and imprecise algorithms. The theory relies on finite ε-size neighborhoods to define ε-topological operations, which are then used to formulate the concept of ε-regularity and, finally, the notion of ε-solidity. The theory contains the classical theory of solid modeling as a special (though unrealistic) case.

Readers familiar with mathematical morphology [34] may notice some similarity with morphological algebra that is typically constructed in terms of operations of dilation and erosion. This similarity is superficial. For details see [24], where we explain why morphological operations are not adequate for the task and explore their relationship to the ε-topological operations and to the proposed theory.

2.1 ε-Topological Operations

If we accept that both geometric data and algorithms are inherently imprecise, we have no choice but to reconsider the very foundations of the solid modeling theory. Traditionally, solidity is formulated in terms of the classical topological concepts and operations of closure k, interior i, and boundary ∂ that are interpreted in terms of infinitesimal size neighborhoods. In contrast, inaccuracy of data and finite resolution of the algorithms imply that the neighborhoods of every point may be represented only up to some finite size. This in turn requires redefining the usual topological operations.

Specifically, we propose the ε-topological counterparts of the classical topological operations: *ε-closure* k_ε, *ε-interior* i_ε, and *ε-boundary* ∂_ε, where ε is a non-negative real number. Intuitively, ε corresponds to the maximum algorithm precision and/or maximal data precision. In the following definitions, we refer to a metric space as (W, d), where W is a non-empty set and d is a suitable distance function. In a metric space, $B(x, r)$ denotes the open ball about point x of radius r.

Definition 2.1. Given a subset X of a metric space, a point x is said to be in the *ε-closure* of X, denoted $k_\varepsilon(X)$, if for every $r > \varepsilon$, $B(x, r) \cap X \neq \emptyset$, where ε is a non-negative real number.

Definition 2.2. Given a subset X of a metric space, a point x is said to be in the *ε-interior* of X, denoted $i_\varepsilon(X)$, if there exits an open ball about x of radius $r > \varepsilon$ such that $B(x, r) \subseteq X$, where ε is a non-negative real number.

Definition 2.3. Given a subset X of a metric space, a point x is said to be in the *ε-boundary* of X, denoted $\partial_\varepsilon(X)$, if x is in both $k_\varepsilon(X)$ and $k_\varepsilon c(X)$, where $c(X)$ denotes the complement of a set.

The above ε-topological operations are illustrated in Fig. 1. For $\varepsilon = 0$, these operations correspond to the usual classical topological operations, and in this sense, they are generalizations of the corresponding definitions in the general (point-set) topology [18, 19]. But for $\varepsilon > 0$, additional points are added to or subtracted from closure or interior respectively, and the boundary of the set is "thickened" by the ball of radius ε. Notice that operations k_ε and

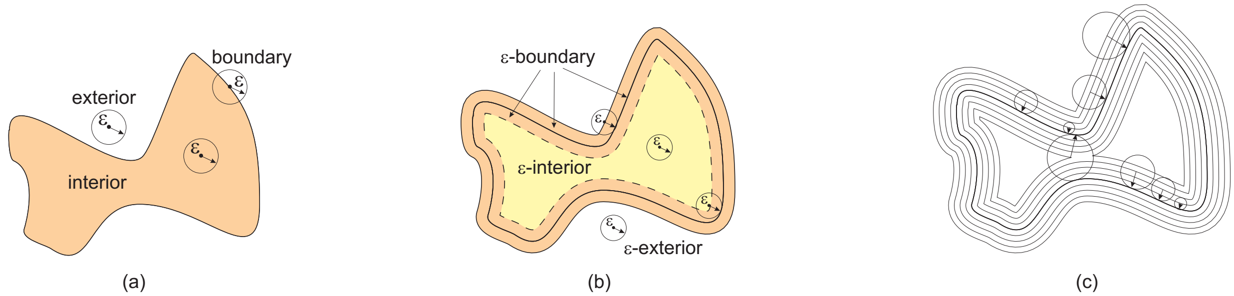

Figure 1: Classical topological operations and the corresponding ε-topological operations. (a) Classical topological operations are defined using infinitesimal $\varepsilon = 0$ neighborhoods. (b) ε-topological operations are defined using finite size neighborhoods. (c) Both ε-interior and ε-exterior decrease, and ε-boundary grows as ball radius ε increases.

i_ε define sets that are closed and open in the usual metric topology with $\varepsilon = 0$. It is not difficult to show that the ε-topological operations inherit and preserve many of the classical properties, as illustrated by the theorems below. In what follows, $c(X)$ denotes the complement of X.

Theorem 2.4. *A point x is in the complement of $k_\varepsilon(X)$ if and only if there is an open ball about x of radius $r > \varepsilon$ such that $B(x, r) \cap X = \emptyset$.*

Proof. If $B(x, r) \cap X = \emptyset$ with $r > \varepsilon$, then it is also true that there is a $B(x, r) \cap k_\varepsilon(X) = \emptyset$ with $r > 0$. Therefore, x is not in $k_\varepsilon(X)$. Conversely, if x is in the complement of $k_\varepsilon(X)$, there is a $B(x, r) \cap k_\varepsilon(X) = \emptyset$ with $r > 0$, and therefore $B(x, r) \cap X = \emptyset$ with $r > \varepsilon$. □

Theorem 2.5. $i_\varepsilon(X) = ck_\varepsilon c(X)$.

Proof. If x is a point in $ck_\varepsilon c(X)$, then there is a $B(x, r)$ of $r > \varepsilon$, $B(x, r) \cap c(X) = \emptyset$, or to say $B(x, r) \subseteq X$, so from Definition 2.2, x is a point in $i_\varepsilon(X)$. Conversely, if x is a point in $i_\varepsilon(X)$, then there is a $B(x, r)$ of $r > \varepsilon$, $B(x, r) \subseteq X$, so $B(x, r) \cap c(X) = \emptyset$, so from Theorem 2.4, x is a point in $ck_\varepsilon c(X)$. □

Corollary 2.6. $ci_\varepsilon(X) = k_\varepsilon c(X)$, $k_\varepsilon(X) = ci_\varepsilon c(X)$.

Similarly, it can be shown that many (but not all[1]) classical theorems are preserved under these generalized topological operations [10]. From the above definitions, a point cannot be in both ∂_ε and i_ε. Thus, every set $k_\varepsilon(X)$ is partitioned into its ε-interior and a 'thickened' boundary as $k_\varepsilon(X) = \partial_\varepsilon(X) \cup i_\varepsilon(X)$. It follows immediately that for any set X,

$$i_\varepsilon(X) \subseteq X \subseteq k_\varepsilon(X). \tag{1}$$

The set of points that are *not* in $k_\varepsilon(X)$ is called *ε-exterior* of set X and will be denoted $e_\varepsilon(X)$. It can be also defined directly following Theorem 2.4 using open ball. Since

$$e_\varepsilon(X) \cup \partial_\varepsilon(X) \cup i_\varepsilon(X) = W, \tag{2}$$

we can say that any set $X \subseteq W$ induces a partition of the space W under the ε-topological operations.

Consider what happens to the induced partition of space W under different values of ε (Fig. 1(c)). For a given set X, a larger value of ε will result in a shrunk ε-interior $i_\varepsilon(X)$ and a grown ε-closure $k_\varepsilon(X)$. As a result, the ε-boundary will thicken further as well. We can summarize this concisely by the theorem whose proof follows directly from the above definitions.

Theorem 2.7. *For a given subset X of a metric space, and non-negative numbers ε_1 and ε_2, if $\varepsilon_1 \geq \varepsilon_2$, then*

$$i_{\varepsilon_1}(X) \subseteq i_{\varepsilon_2}(X), e_{\varepsilon_1}(X) \subseteq e_{\varepsilon_2}(X), \partial_{\varepsilon_1}(X) \supseteq \partial_{\varepsilon_2}(X). \tag{3}$$

[1] Notably, the classical property $k(k(X)) = k(X)$ does not hold for k_ε operation when $\varepsilon > 0$.

Theorem 2.7 implies that i_ε and e_ε are monotonically decreasing functions of ε: as ε decreases, points can only be added to the interior $i_\varepsilon(X)$ and exterior $e_\varepsilon(X)$, while they are removed from the shrinking boundary $\partial_\varepsilon(X)$. As ε approaches 0, e_ε, i_ε, and ∂_ε approach the classical exact sets of exterior, interior, and boundary respectively.

2.2 ε-Regular Sets

The classical models of solidity postulate that a solid is a set of points with dimensionally homogeneous interior and well defined boundary. This notion can be captured by requiring every valid computer representation correspond to at least one well defined *regular* set X:

- closed regular $X = ki(X)$ [27], or
- open regular $X = ik(X)$ [3].

Both models correspond to dimensionally homogeneous (without cracks or dangling pieces) sets with tight boundaries, the main difference being that closed regular sets include their boundaries while open regular do not. A common characterization of solidity in both models is that a neighborhood of every boundary point contains points in the solid's interior as well as points in its exterior.

As we argued above, identifying a regular set in the presence of errors is not always feasible, practical, or even desirable. Representation of X is usually generated from user inputs, numerical computations, approximations, and physical samplings – all of which contain some *errors*. These data errors in the representation of X usually violate the validity conditions (numerical, topological, combinatorial) implied by the properties of regular sets. For example, Figure 2 shows typical 'invalid' 2-dimensional boundary representations of a simple rectangle; as with most boundary representations, the vertices, edges, and faces represent geometric information only approximately and redundantly. Depending on how the boundary is constructed, it also may include small imperfections: isolated and dangling pieces, voids, gaps. Applying the notions of (exact) closure, interior, and boundary to such representations and requiring regularity does not really make sense, because strictly speaking, all such representations are invalid. Hence, the key question is: *What is a proper model of solidity in terms of ε-topological operations* that would tolerate errors of size less than ε, such as those shown in Fig. 2? Below, we answer this question by proposing notions of ε-regularity and ε-solidity that subsume the corresponding classical notions.

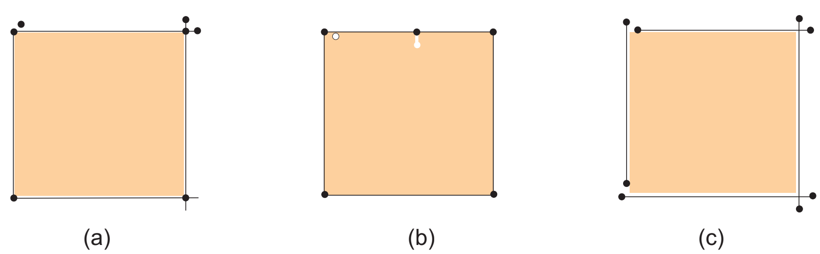

Figure 2: A theory of ε-solidity must tolerate imperfections of size less than ε near the theoretical boundary of a solid: (a) dangling edges or isolated points; (b) small cracks and voids; (c) misaligned or redundant vertices and edges.

Intuitively, it should be apparent that a proper definition of regularity must not eliminate but *tolerate* errors and imperfections near the set boundary. The definition should be consistent with the classical notions of regular sets, and in fact the generalization is relatively straightforward. We *postulate* that a *known* set of X should be deemed ε-regular if *an ε-neighborhood of every boundary point $p \in \partial_0(X)$ contains points in the set interior $i_0(X)$ and points in the set exterior $e_0(X)$*. The challenge is to formulate this postulate in terms of the ε-topological operations in a manner that is consistent with the classical notions of regularity.

Let us first consider what this means when $\varepsilon = 0$. By definition, $i_0(X)$ is the largest open set contained in X, and $k_0(X)$ is the smallest closed set that contains X, which means that for any set X,

$$i_0(X) \subseteq X \subseteq k_0(X). \tag{4}$$

Neither one of these bounding sets are guaranteed to be homogeneous in dimension: $i_0(X)$ may contain voids and cracks, while $k_0(X)$ may include dangling pieces and isolated points. The homogenization (or regularization) is achieved by a second topological operation that grows the lower and shrinks the upper bounding sets respectively: $k_0i_0(X)$ is the smallest closed regular set containing $i_0(X)$, and $i_0k_0(X)$ is the largest open regular set contained in $k_0(X)$. The regularization effectively reverses the set relationship (4) to

$$i_0k_0(X) \subseteq X \subseteq k_0i_0(X). \tag{5}$$

It should be clear that any open or closed regular set X satisfies the inequality (5) whose essence is to combine the two classical definitions into a single definition of an 0-regular set. But other sets satisfy the above inequality as well, because it allows for imperfections (missing portions or isolated points) in the boundary $\partial_0(X)$. For example, a unit sphere with half of its boundary missing would satisfy (5) and be considered 0-regular.

However, the two homogeneous bounds (5) are so tight that they are not realistic, because no set X satisfying (5) may contain even minor imperfections in its interior or exterior, such as those shown in Fig. 2. To say that imperfections are tolerated in the interior of the set X *within distance* ε from the boundary amounts to a statement that X contains $i_\varepsilon k_0(X)$; similarly, when $k_\varepsilon i_0(X)$ contains the set X, exterior imperfections within ε from the boundary are tolerated. This motivates our first attempt at definition of ε-regularity for a *known* set X.

Definition 2.8. A subset X of a metric space is *ε-regular* if, for a given non-negative real number ε,

$$i_\varepsilon k_0(X) \subseteq X \subseteq k_\varepsilon i_0(X). \tag{6}$$

Informally, the above definition assumes that we can determine the closure and interior of X exactly, but we allow errors within distance ε near the boundary. The concept is illustrated in Fig. 3 showing that imperfections within ε of the boundary in Fig. 2 are "covered" by either growing $i_0(X)$ by k_ε as in Fig. 3(a), or by shrinking $k_0(X)$ by i_ε as in Fig. 3(b). Thus, in contrast to the classical definitions, under the proposed notion of ε-regularity, these representations of rectangles are considered to represent ε-regular sets.

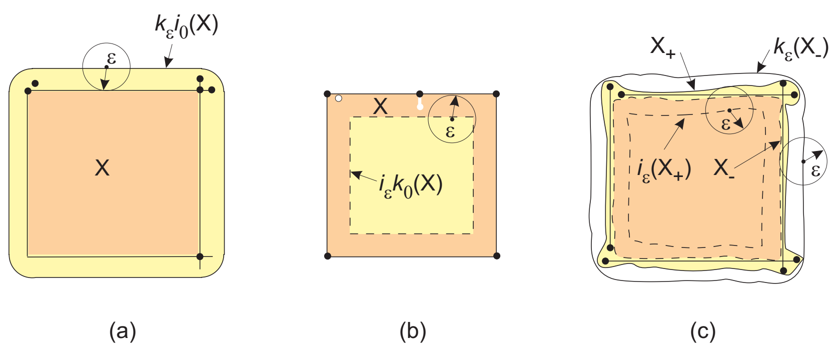

Figure 3: ε-regular sets and intervals tolerate imperfections of size less than ε near the set boundary: (a) growing $i_0(X)$ to cover dangling edges and isolated points; (b) shrinking $k_0(X)$ to cover small cracks and voids; (c) an interval $[X_-, X_+]$ covers the imperfection by $i_\varepsilon(X_+)$ and $k_\varepsilon(X_-)$.

By Definition 2.8, ε-regularity of a set depends on the size of chosen ε. Classical regular sets (open and closed) clearly satisfy the definition in the special case when $\varepsilon = 0$, with one side of (6) becoming an equality. Inequality is crucial in the general case, because it corresponds to the notion of tolerant modeling near the boundary of the set. By definition, increasing the value of ε corresponds to shrinking $i_\varepsilon k_0(X)$ and expanding $k_\varepsilon i_0(X)$. Thus, if X is ε-regular, it is also ε_1-regular for any $\varepsilon_1 \geq \varepsilon$. We observe that every bounded set (or rather every set containing bounded 'imperfections') with non-empty interior is ε-regular for some sufficiently large value of ε. Normally, we are interested in the smallest value of ε for which X is ε-regular.

2.3 ε-Regular Intervals

We now consider more realistic situations where we may not be able to compute the interior $i_0(X)$ and closure $k_0(X)$ of a set X exactly, but only some bounding sets can be determined – either as input or as a result of another approximate computation. In what follows, we will refer to the bounding set X_- as *inner*, and X_+ as *outer*, assuming the inner is open and the outer is closed. An example of inner and outer bounding sets is shown in Fig. 3(c). In this sense, the interior $i_0(X)$ and the closure $k_0(X)$ are the tightest bounds computable for any given set X. The inner and outer sets form a *set interval*, $[X_-, X_+]$ — *the class of sets* $\{X\}$ such that $X_- \subseteq X \subseteq X_+$. Then Definition 2.8 generalizes in a straightforward fashion by replacing $k_0(X)$ and $i_0(X)$ respectively with arbitrary outer X_+ and inner X_-:

Definition 2.9. A set interval $[X_-, X_+]$ is ε*-regular* if, for a given non-negative real number ε,

$$i_\varepsilon(X_+) \subseteq X_- \subseteq X_+ \subseteq k_\varepsilon(X_-). \tag{7}$$

Technically, the test for ε-regularity of an interval depends on two separate conditions illustrated in Fig. 4: $i_\varepsilon(X_+) \subseteq X_-$ requires that when outer X_+ is shrunk by a ball of size ε, it fits inside the inner; similarly $X_+ \subseteq k_\varepsilon(X_-)$ requires that inner X_- grown by a ball of size ε contains outer X_+. For example, the rectangle in Fig. 3(c) is not regular in the classical sense, but is ε-regular as an interval $[X_-, X_+]$. Once again, if the interval is ε-regular for any particular value of ε, then it must also be ε-regular for any greater value of ε, but not necessarily for the smaller. In

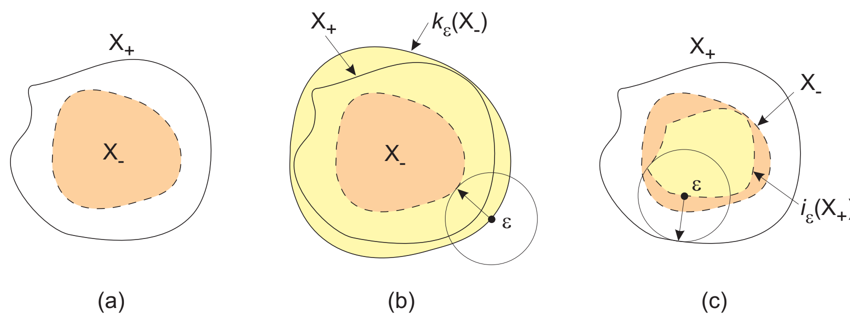

Figure 4: A set interval becomes ε-regular if the ε is big enough: (a) set interval $[X_-, X_+]$; (b) $X_+ \subseteq k_\varepsilon(X_-)$; (c) $i_\varepsilon(X_+) \subseteq X_-$.

practice, we are usually interested in the smallest value of ε for which a given interval $[X_-, X_+]$ is ε-regular. On one hand, ε must be large enough to cover and hide the imperfections near the boundaries of the sets in the interval; on the other hand, it is a measure of closeness between the sets in the interval. Informally, we would like to say that all sets in an ε-regular interval $[X_-, X_+]$ are within ε of each other. Formally, we showed in [24] that the Haudorff distance between the inner and outer of an ε-regular interval, as well as between their respective boundaries and complements, is bounded by ε.

Definition 2.9 for a set interval is written in the same form as Definition 2.8 for a set instance, in order to emphasize their common structure. In fact, it is easy to show that Definition 2.8 is a special case of Definition 2.9, by recalling that any set X is contained in the interval $[i_0(X), k_0(X)]$.

Theorem 2.10. *A set X is ε-regular iff the interval $[i_0(X), k_0(X)]$ is ε-regular.*

Proof. From Definition 2.8, we have $k_\varepsilon i_0(X) \supseteq k_0(X)$, and $i_0(X) \supseteq i_\varepsilon k_0(X)$. Thus, let $X_- = i_0(X)$ and $X_+ = k_0(X)$. For the interval $[X_-, X_+]$, we have $k_\varepsilon X_- \supseteq X_+$, and $X_- \supseteq i_\varepsilon X_+$, which is an ε-regular interval. Conversely, if $[i_0(X), k_0(X)]$ is ε-regular, then from Definition 2.9, $k_\varepsilon i_0(X) \supseteq k_0(X)$, $i_0(X) \supseteq i_\varepsilon k_0(X)$. Thus, X is ε-regular. □

In other words, we really need only Definition 2.9 of ε-regular interval, because it subsumes Definition 2.8 of an ε-regular set. Henceforth, it should be understood that the term 'ε-regular interval' also applies to ε-regular set instances. Furthermore, it is easy to see that every set instance X in an ε-regular interval $[X_-, X_+]$ is also ε-regular. This is reasonable and should be expected, since every such set interval represents an equivalence class of sets that are not distinguishable beyond the inner and outer bounds of the interval. In fact, if we define a *subinterval* $[Y_-, Y_+]$ of $[X_-, X_+]$ as *the class of sets* $\{Y\}$ such that $Y_- \subseteq Y \subseteq Y_+$, with the inner $Y_- \supseteq X_-$ and the outer $Y_+ \subseteq X_+$, we can make an even stronger claim:

Theorem 2.11. *Any subinterval* $[Y_-, Y_+]$ *of an* ε*-regular interval* $[X_-, X_+]$ *is also* ε*-regular.*

Proof. By definition, X_-, Y_- are open, and X_+, Y_+ are closed sets, with $Y_- \supseteq X_-$ and $Y_+ \subseteq X_+$. Since $k_\varepsilon(X_-) \supseteq X_+$, then $k_\varepsilon(Y_-) \supseteq k_\varepsilon(X_-) \supseteq X_+ \supseteq Y_+$; similarly, since $X_- \supseteq i_\varepsilon(X_+)$, then $Y_- \supseteq X_- \supseteq i_\varepsilon(X_+) \supseteq i_\varepsilon(Y_+)$. Thus, $[Y_-, Y_+]$ is ε-regular. □

This result is of paramount *practical* significance, because it allows to verify regularity of an interval $[Y_-, Y_+]$ even when the interval itself is not computable by testing a larger containing interval $[X_-, X_+]$ that is computable. In particular, Fig. 5 shows that any set X contained in an ε-regular interval must be ε-regular. This statement is conservative in a sense that X may be ε-regular with even a smaller value of ε.

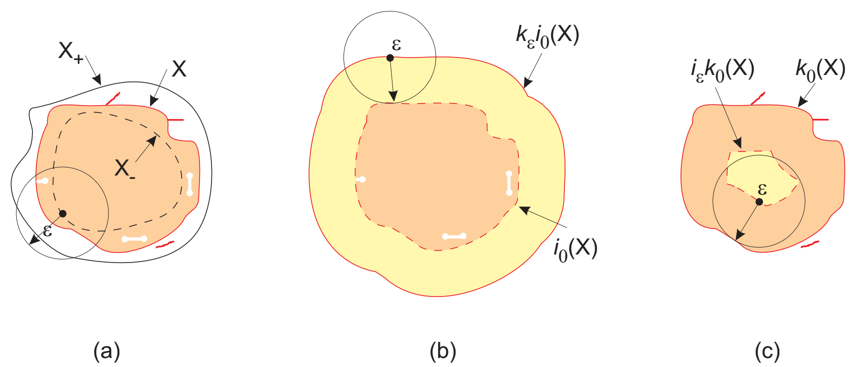

Figure 5: Any set contained in an ε-regular interval is ε-regular with the same or smaller value of ε: (a) set instance X with dangling pieces and inner cracks, contained in an ε-regular set interval; (b) $X \subseteq k_\varepsilon i_0(X)$; (c) $i_\varepsilon k_0(X) \subseteq X$.

2.4 ε-Solidity

The ε-regularity extends the notion of solid homogeneity to set intervals and, as such, is a necessary condition for the notion of ε-solidity. Note however, that the inner X_- and the outer X_+ of an ε-regular interval do not need to be dimensionally homogeneous sets. It is also customary to require that every solid is bounded and has a non-empty interior. These statements translate respectively into additional constraints on set intervals requiring that the inner X_- is non-empty and the outer X_+ is bounded.

Definition 2.12. An ε*-solid* is an ε-regular set interval $[X_-, X_+]$ with non-empty X_- and bounded X_+.

Given any computer representation of a set, its interior and closure can be computed only within some precision δ. In this case, $i_\delta(X) \subseteq X \subseteq k_\delta(X)$, but since X is really not known, it is more reasonable to consider interval $[i_\delta(X), k_\delta(X)]$ in place of the set. If $i_\delta(X)$ is not empty and $k_\delta(X)$ is bounded, this interval is an ε-solid if it is ε-regular, for a particular choice of ε. Following Definition 2.9, the later requires satisfaction of two conditions:

$$i_\varepsilon k_\delta(X) \subseteq i_\delta(X), \qquad \text{and} \qquad k_\delta(X) \subseteq k_\varepsilon i_\delta(X). \tag{8}$$

The choice of ε is critical. Smaller ε is preferable because it corresponds to a more accurate representation, but if we choose $\varepsilon < \delta$, then the interval *cannot* be regular because $k_\varepsilon i_\delta(X) \subseteq X \subseteq i_\varepsilon k_\delta(X)$. The regularity conditions (8) can only be satisfied starting with some $\varepsilon \geq \delta$. At this point, we would have established that the interval $[X_-, X_+]$ is indeed an ε-solid according to Definition 2.12. Furthermore, the following theorem is an immediate corollary to Theorem 2.11 and implies that the set X itself is also an ε-solid.

Theorem 2.13. *Any subinterval* $[Y_-, Y_+]$ *of* ε*-solid* $[X_-, X_+]$ *is* ε*-solid.*

Proof. From Theorem 2.11, any subinterval in $[X_-, X_+]$ is ε-regular. Since the inner bound X_- is non-empty, so is the corresponding inner bound of the subinterval Y_-; the boundedness of the outer set X_+ implies the boundedness of Y_+. Thus the subinterval $[Y_-, Y_+]$ is an ε-solid. □

For example, if we can establish that any $[i_\delta(X), k_\delta(X)]$ is ε-solid, then our formulation implies that $[i_0(X), k_0(X)]$, and hence X itself, is also ε-solid – even though X may contain errors or may not be known. For a specific given value of $\varepsilon \geq \delta$, X may or may not be ε-solid. However, unless set X is unbounded or its interior $i_\delta(X)$ is empty, it is an ε-solid for *some* sufficiently large value of ε. An essentially identical procedure may be used to determine if an arbitrary interval $[X_-, X_+]$ is ε-solid.

The above notions of ε-regularity and ε-solidity recognize explicitly that the closure and interior of any set may be determined only up to some limited precision δ and may be tested for regularity only within some resolution $\varepsilon \geq \delta$. Thus, a given ε-solid X now may or may not be solid in the classical sense, depending on the specific values of δ and ε. This is reasonable, since in practice we are not likely to have exact representations and computations of X.

3 Validity of ε-Solid Representations

The above definitions and theorems explicitly acknowledge that topological properties of geometric data may be represented or computed only within some finite precision. The challenge is to build on these definitions, in order to formulate the notion of validity of solids in the presence of errors and limited precision, and to develop practical algorithms that implement these definitions.

3.1 Accuracy of Data and Precision of Algorithms

Representations of geometric objects are stored on a computer and queried by algorithms. Below we will distinguish between the *accuracy* of a geometric representation and the *precision* of geometric queries on this data. The accuracy of any point of the geometric data is measured by a λ-radius of the uncertainty ball about that point. If the data comes from the ideal source and is represented exactly, $\lambda = 0$; if the source of the data is only approximate or the data underwent an approximate conversion, $\lambda > 0$. Unfortunately, accuracy of geometric data is not commonly archived.

Most geometric algorithms and queries on geometric data sooner or later reduce to a finite number of Point Membership Classification (PMC) tests [35]. For a given point p and a set X, PMC returns IN, ON or OUT depending on whether p belongs to the interior, boundary, or exterior of X respectively. The classical semantics of PMC in solid modeling assumes an ability to compute an arbitrary small neighborhood of point p in order to identify the interior, exterior and boundary of X. However, all practical implementations of PMC algorithms have finite precision and always rely on finite size neighborhoods implied by floating point roundoff errors and approximations. Even when a set X is represented exactly ($\lambda = 0$) using a finite collection of geometric primitives, PMC tests near the boundary of the primitives have limited precision, and return ON not only for the points on the boundary but also for the points that are near the boundary. Thus, when we say that a geometric representation can be queried with precision δ, we specifically refer to the precision of a PMC_δ test that is defined in terms of the ε-topological operations as:

$$\text{PMC}_\delta(p, X) = \begin{cases} IN & \text{if } p \in i_\delta(X), \\ OUT & \text{if } p \in e_\delta(X), \\ ON & \text{if } p \in \partial_\delta(X). \end{cases}$$

In other words, PMC_δ is just an operational definition of the partition of space (Equation (2)) induced by any set X under the δ-topological operations. It is a straightforward generalization of the classical PMC test that reflects the role of precision in practical implementations of the test. The classical PMC is just a special exact case of the above test with $\delta = 0$. Informally, PMC_δ classifies a point p as IN or OUT with respect to a given set X if it is farther than δ from the set boundary, otherwise the point is classified as ON. Since X itself is usually represented with some finite accuracy λ, PMC_δ test makes sense only when its precision is greater than accuracy of the data, i.e. under all conditions, we must have $\delta \geq \lambda$ – irrespective of the semantics of the representation scheme.

The precision of PMC$_\delta$ is a major limiting factor in deciding whether a geometric data represents a valid ε-solid or not. Recall from Section 2.4 that an interval $[i_\delta(X), k_\delta(X)]$ may be an ε-solid *only* when $\varepsilon \geq \delta$. It may be tempting to identify δ and ε and treat them as one constant, but the distinction is semantically important, as illustrated by the following examples.

Consider a standard query on a CSG representation [38]. Figures 6(a)-(c) show the classical regularized intersection operation that assumes exact representations of primitive sets A and B and ability to compute arbitrarily small neighborhoods. In practice, however, depending on the precision of PMC$_\delta$ algorithm, a point may be classified with respect to each primitive only within some precision δ. The classifications from individual primitives are combined according to the regularized set operations (Fig. 6(d)), but the regularized combination operation depends critically on explicit representation and analysis of the neighborhood of the point. As shown in Fig. 6(e), the size of the neighborhood ε must be greater or equal to δ, or regularization may fail, producing results that are inconsistent with the semantics of the exact regularized set operations.

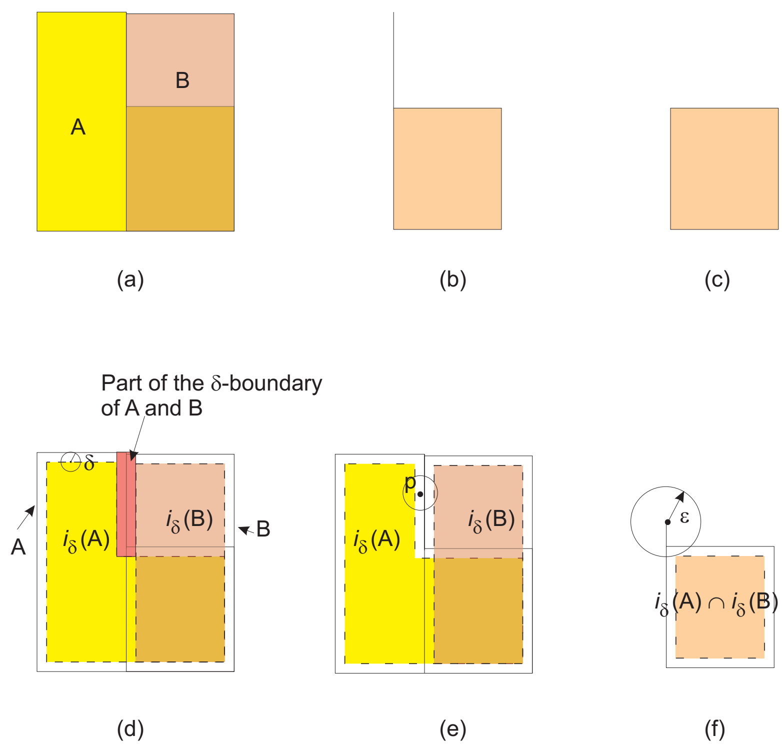

Figure 6: Comparison of the theoretically exact PMC on CSG representation (top row) and PMC$_\varepsilon$ recognizing limited precision of PMC$_\delta$ against individual primitives (bottom row). (a) Two 2D closed regular sets A and B, $A = k_0 i_0(A)$, $B = k_0 i_0(B)$. (b) Standard intersection of A and B, $A \cap B$ allows dangling “face of any size. (c) Classical regularized intersection of A and B gives $k_0 i_0(A \cap B)$. (d) Testing interior and boundary sets of A and B by finite precision δ-PMC. (e) Neighborhood of point p has to be larger than δ to include points from $i_\delta(A)$ and $i_\delta(B)$ for regularization. (f) The ε-regularization removes points which are ε far from $i_\delta(A) \cap i_\delta(B)$.

The distinction between δ and ε is even more pronounced for boundary representations. Consider the boundary representation of a rectangle in Fig. 9(a). The faces (in this case, the line segments) may or may not be represented exactly (i.e. λ may or may not be 0), but their connectedness may be established only within precision $\delta \geq \lambda$. Depending on selected value of ε, the connected set itself may or may not be a boundary of ε-solid, but under all conditions we must have $\varepsilon \geq \delta$ (Fig. 9(b)).

3.2 Validity and Repair of Representations

The ε-solid model is a generalization of the classical r-set solid model and requires a significant revision of the modeling paradigm proposed in [27], as shown in Fig. 7. The extended mathematical *modeling space* is the space of all ε-solids, which includes all closed and open regular sets. For any given representation scheme, the *representation space* is the space of representations – pairs (R_λ, PMC$_\delta$), where R_λ is a geometric data represented with accuracy λ and PMC$_\delta$ is a point classification procedure with precision $\delta \geq \lambda$. The *representation scheme* itself is a mapping from the space of ε-solids to representation pairs (R_λ, PMC$_\delta$). In this updated modeling paradigm, the definition of representation *validity* reflects more accurately how solids are handled in modern modeling systems.

Definition 3.1. Representation (R_λ, PMC$_\delta$) is *valid* if application of PMC$_\delta$ on R_λ induces an ε-solid.

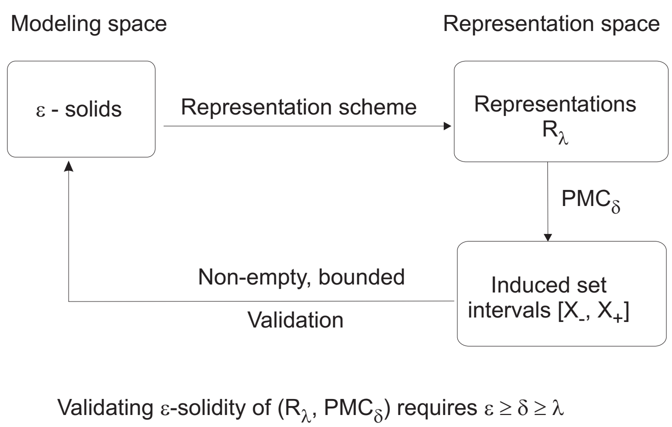

Figure 7: Modeling and representation spaces for ε-solids with limited data accuracy λ and algorithm precision δ.

This notion of validity should support systematic development of provably correct algorithms and will allow us to validate and verify existing and proposed methods. In classical solid modeling, validation of boundary representation requires proper connectivity of the vertices, edges, faces, and verification that the shells of a boundary representation are two-manifolds [13, 27, 31]. The key underlying assumptions are that the metric conditions of such a model are required to be exact, and the queries of intersection tests are exact too. In practice, both assumptions are not true for applications such as robust geometric computations and geometric data translation, i.e. geometric carriers (curves, surfaces) improperly overlap or do not match "closely enough," numerical tolerances are used in queries. Various surgical operations are adopted in the validation techniques with an attempt to maintain "valid" solids via geometric and topological modifications of the boundary representation (re-intersecting edges, removing small edges and faces, merging adjacent vertices and edges). As a demonstration, it is easy to argue, *based on empirical evidence and assumed theoretical foundations*, that many currently adopted validation procedures may not be sufficient and commonly used geometric repairs may not be necessary for maintaining solidity when the boundary representation and/or corresponding quires are not exact.

Figure 8(a) shows a tessellated boundary representation of a circle. This is a valid solid, but every vertex has a PMC tolerance zone of size δ, and the zones of adjacent vertices overlap as shown. One popular technique used for robust geometric computations merges the adjacent vertices whenever their zones overlap into a single vertex with a larger zone [33] (healing techniques in data translation also use the same idea to remove small features). In this example, the process will result in a single vertex with a tolerance zone covering the whole polygon, suggesting that the object is not a valid polygon. But as shown in Fig. 8(b), the polygon has a well-defined non-empty δ-interior and a bounded δ-closure. Thus, a suitable PMC$_\delta$ test on the representation gives an ε-regular set interval for some number ε. In other words, this particular validation procedure is not sufficient for verifying ε-solidity. Another familiar and simple example is shown in Fig. 9(a). In this case, a typical boundary-based validation procedure would determine that the representation is invalid, because the adjacent edges don't intersect at common vertices. A healing procedure would usually be invoked to re-intersect the edges, in order to get more precise vertices. But it should be clear from

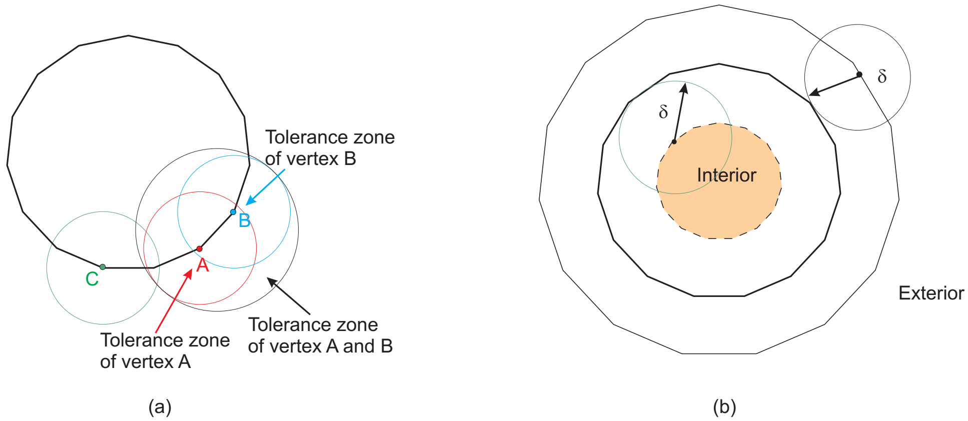

Figure 8: Solid validation procedure based on boundary representation is not sufficient under indicated tolerances: (a) vertex merging algorithm indicates that the representation is a single point, and hence is invalid; (b) non-empty i_δ and bounded k_δ indicate a valid boundary representation of a solid.

Fig. 9(b) and the discussion in sections 2.4 and 3.1, that with suitable PMC$_\delta$ we can still induce the interior and exterior of the rectangle — even though the connectivity of adjacent vertices and edges is not guaranteed. In other words, in this case, this is a valid boundary representation of an ε-solid and no additional geometric repairs are necessary for maintaining ε-solidity, as long as the value of ε is sufficiently small to support the intended application.

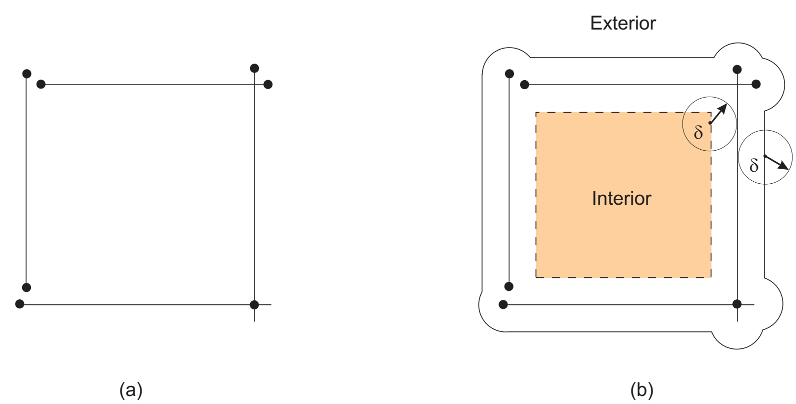

Figure 9: Repair of boundary representation is not necessary to assure validity under tolerances: (a) common errors in boundary representation indicate invalid object; (b) a suitable choice of PMC$_\delta$ procedure can classify points of the interior and exterior in the presence of errors.

4 Conclusions

4.1 Summary

This research started several years ago [23] with a relatively modest goal of formulating and solving several specific problems in geometric data translation (see [25] for specific discussion on geometric data translation). Since the classical theory of solid modeling assumes exact geometric data and algorithms, it became clear that it must be substantially generalized before it could be applied to such problems. This paper proposes such a generalization, based on the observation that topological properties of sets may be represented and computed only within some finite precision.

The formulated notions of ε-regularity, ε-solidity, and PMC$_\delta$ capture more realistically the practices and the recognized limitations of geometric and solid modeling. Importantly, the new theory subsumes the classical solid modeling formulations as a special (but not very realistic) case of $\varepsilon = 0$. In this sense, the proposed formulation of the problem certainly does not preclude exact representations and computations whenever such are feasible and practical. Nor is the theory limited by any particular choice of exact constants δ or ε. For example, it may be convenient to visualize and identify the δ-topological operations with the classical offsetting operations in solid modeling [30]; but in practice, PMC$_\delta$ is rarely implemented for any fixed value of δ; the discussion and observations in Section 3 still apply following Theorem 2.11, as long as the interval $[X_-, X_+]$ is computed conservatively with $i_\delta(X) \subseteq X_-$ and $k_\delta(X) \supseteq X_+$. Several common examples of this situation are discussed in [24].

We also demonstrated that widely accepted methods for validating boundary representations may not suffice for maintaining ε-solidity in the presence of numerical inaccuracies, and argued by example that geometric healing procedures may be avoided in many common situations.

In a longer term, the proposed revision of the classical solid modeling paradigm recognizes explicitly that accuracy λ of representation R and precision δ of PMC algorithms cannot and should not be considered separately from each other. Our observations and formulation point the way to improved redesign of both data structures and algorithms for solid modeling that explicitly recognize the distinct semantic roles of three physical constants: accuracy λ, precision δ, and solidity tolerance ε. The need for such a redesign has been apparent for some time, as witnessed by numerous efforts to deal with accuracy problems in STEP models and translations. Because our formulation does not require existence of exact valid objects (such as r-sets or manifolds), a number of robustness or validation problems may be easier than they appear. For example, Fig. 6(f) shows that a PMC$_\varepsilon$ does not need to resolve dangling boundaries or other imperfections that are within ε of the interior i_δ of the solid. Thus, properly redefined ε-regularized set operations can be used to keep track and control the errors near solid's boundary.

4.2 Broader Implications

A significant feature of the proposed formal framework is that it is mostly representation free, in a sense that it does not assume any particular representation, approximation, or discretization of the represented pointsets. Application of the proposed formulation in specific representational problems should shed useful interpretations and establish relationship between seemingly unrelated techniques. For example, Delaunay-based solid reconstruction methods [2] and voxel-based approximation techniques [22, 32] appear to be directly related to problems of ε-solidity.

The concepts of ε-topological operations and ε-regularity may be also useful in formalizing semantics of geometric dimensioning and tolerancing (GD&T). Previous formulations proposed that a toleranced mechanical part is a class of regular sets [28] that contains an exact nominal set, as well as special perfect bounding (least and maximal) elements. Such a class of sets itself must satisfy additional metric and regularity conditions [8]. Generating and testing sets for membership in a tolerance class remains an open problem, particularly because the proposed definitions do not take into account limited resolution of inspection. In contrast, our approach using ε-topological operations may be more effective, because it would not require (but would allow) knowledge of any perfect regular elements while explicitly taking into account limited precision of representation and inspection.

A number of theoretical issues remain open. Accepting ε-topological operations and ε-solidity as foundations for solid modeling requires non-trivial revision of the usual concepts and operations. These include ε-continuity, ε-homeomorphism, and ε-regularized set operations. For example, definitions of ε-regularized set operations may appear straightforward, based on the definitions in this paper. However, such operations may not posses desired algebraic properties, such as obeying the distributive law. Many of the difficulties stem from recognition that an ε-solid is really a set interval. It is reasonable to expect that interval analysis [20] must form a basis for any algebraic system of ε-solids, but substantial enhancements are likely to be needed before such a system is useful for solid modeling. Such an algebraic system should have applications beyond solid modeling, e.g. in the area of robust geometric computations. Connections with computable analysis [39], constructive analysis [6], and domain theory [1] are also of considerable interest.

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