Research in Engineering Design manuscript No.
(will be inserted by the editor)

A class of forms from function: the case of parts moving in contact

Horea Ilieş[1], **Vadim Shapiro**[2]

[1] Ford Motor Company*, e-mail: hilies@ford.com
[2] University of Wisconsin-Madison**, e-mail: vshapiro@engr.wisc.edu

The date of receipt and acceptance will be inserted by the editor

Abstract We consider the general problem of designing mechanical parts moving in contact under the influence of externally applied loads. Geometrically, the problem may be characterized in terms of a *conjugate triplet* which is formed by the two shapes moving in contact and their relative motion. We show that every such triplet belongs to one or more classes of functionally equivalent designs that may be represented uniquely by *maximal* triplets, corresponding respectively to the two largest contact shapes that are guaranteed to contain all other possible solutions to the contact design problem.

In practical terms, the proposed characterization of the contact problem enables the systematic exploration of the design space using fully defined representatives of the functionally equivalent class of parts. Furthermore, such exploration may be performed using standard tools from geometric modeling, and without assuming any particular parametrization that necessarily restrict both the design space and possible computational techniques for exploring feasible designs. Because it supports generation of an essentially unlimited space of design solutions for a given contact problem, the proposed approach is particularly effective at the conceptual design stage.

Key words Design space, shape synthesis, higher pairs, contact, functional equivalence, sweep, unsweep, moving parts.

1 Introduction

1.1 Motivation and scope

Broadly, mechanical design is the transformation, or mapping, from a functional description of a device to a form, or structural description. Every device can be *non-uniquely* described by a set of functions it performs or constraints that it must satisfy which identifies its purpose or intended use. At the same time, there are multiple designs solutions for every functional specification, and, in this sense, each functional specification defines a *class* of functionally equivalent forms.

Higher Kinematic Pairs: Within the group of mechanical constraints, the contact requirements are probably the most complex and interesting ones, receiving much attention in the technical literature. In one of the early studies of contact, Reuleaux [20] introduced the notion of lower and higher kinematic pairs as well as a model for mechanisms consisting of chains of kinematic pairs. A pair usually refers to two bodies moving in

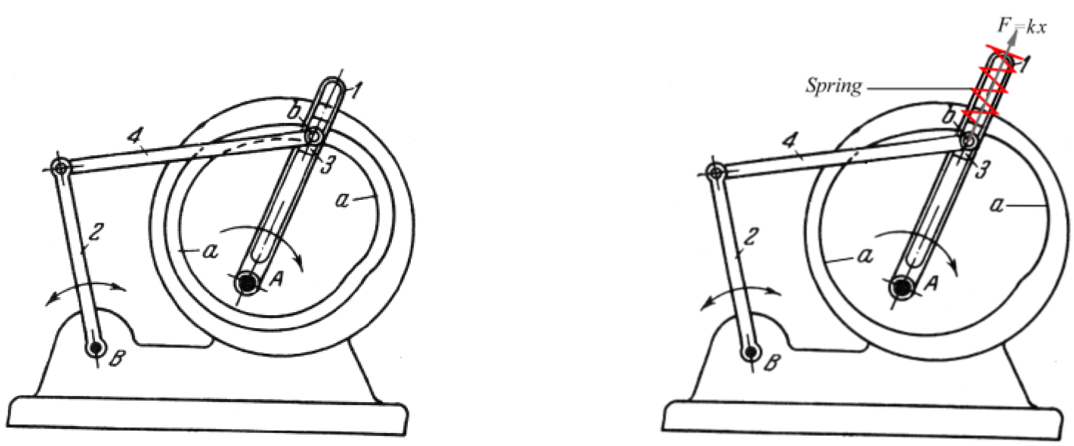
* Please use e-mail as means of contact. Alternate e-mail address: ilies@sal-cnc.me.wisc.edu.
** Spatial Automation Laboratory, Department of Mechanical Engineering, 1513 University Avenue, University of Wisconsin-Madison, 53706 USA.

contact; lower pair refers to the case when the contact takes place over a surface. All other pairs are termed higher pairs and include point, line, curve, or no contact during the motion. In this sense, the higher pairs are substantially more complex than the lower pairs and subsume the latter as special cases. In this paper, we focus on those kinematic pairs that maintain arbitrary type of contact *continuously* throughout the motion.

There are several well known categories of higher pairs that are widely used today in the industry such as various cam-follower mechanisms, gear transmissions, or latching and guiding mechanisms. Traditionally, design of each of these types of higher pairs has relied on specific parametrizations of the design space which, in turn, bred a myriad of specialized design techniques. Generally speaking, the space of design solutions for a given problem is not restricted by the mere existence of a parametrization, but by a parametrization that incompletely describes the design space. For example, the *complete* classification of the lower kinematic pairs into the well known six classes [19] induces parametrizations of any and all lower kinematic pairs and, in this sense, it completely solves the design problem. For higher pairs, no such classification is known, which is why all existing parametrizations only partially describe the design space of the higher pairs. In other words, a common formulation for the shape design of higher pairs that enables a systematic exploration of their design space simply does not exist today. Researchers seem to focus more on the differences between different types of higher pairs rather than on their basic and common property: contact.

Informally, contact of higher kinematic pairs involves two bodies that move relative to each other under externally applied loads, and which come in contact with one another during the motion. In a higher pair, the two bodies share common boundary points, curves, or surfaces during their relative motion. In general, only some of these points of contact would provide contact forces that oppose the penetration, or interference, of the two bodies which withstand the externally applied loads.

Motivating Example: Consider the essentially 2-dimensional mechanism shown in Figure 1(a), reproduced from Artobolevsky's encyclopedia of mechanisms [2]. It contains a link-gear fixed-slot cam mechanism in which the shape of slot $a-a$ is chosen such that roller b slides and rolls along slot $a-a$ according to a specified planar relative motion. Its function is to transform a rotational motion around point A into a prescribed oscillatory motion of link 2. The profile of the slot $a-a$ is designed to recover the desired motion of the link 2. For this mechanism the function can be directly translated into a specification of the one-parameter relative motion between the roller b and slot $a-a$. The transmission of motion is achieved through contact, which essentially classifies the roller b and the slot $a-a$ as a higher pair.



(a) (b)

Fig. 1 A link-gear fixed-slot cam mechanism from [2]. The slider 3, moving along the slot of link 1, has roller b fixed which slides and rolls along fixed slot $a-a$ of special profile. The intermediate link 4 transmits oscillating motion to link 2 about fixed axis B.

Design, however, is an ill-posed problem having multiple solutions satisfying a given function. For example, changing the external forces applied on the roller, as shown in Figure 1(b) makes part of the boundary of the slot $a - a$ redundant. Observe that even if we eliminate here the "inner" boundary of the slot we do not change the function of the mechanism. We may say that the two slot geometries are functionally equivalent, although we did not formally describe what that equivalence means. Furthermore, the new "slot" would have a boundary that is a subset of the original slot, and there are infinitely many slots geometries that would satisfy the same contact function. In fact, the slot shown in Figure 1 (a) contains all other slots satisfying the prescribed contact function, since there are no other points coming in contact with roller b during the given motion.

How do the examples in (a) and (b) differ from a conceptual point of view? How can we characterize the intuitive functional equivalence of the two configurations? Can we use such a characterization to define a notion of functional equivalence that would generate classes of functionally equivalent parts, which, in turn, could be used to systematically generate and explore the space of design solutions for the contact problem? Can we generalize these notions to larger classes of problems? All these questions are fundamental to the development of systematic design procedures that enable systematic studies of the space of design solutions, and will be addressed or considered here.

1.2 Focus and Contribution

We focus on higher kinematic pairs whose shapes maintain contact at *every* instance of their relative motion, because this class of parts represents the majority of the practical higher pairs. The motion itself is assumed to be describable by a one-parameter family of transformations $M(t)$, $t \in [0, 1]$ represented by 4×4 matrices in homogeneous coordinates (see Appendix A). [1]

This paper proposes a systematic approach to identifying, representing, and computing a class of shapes that satisfy the specified geometric requirements of moving contacts. More specifically, we introduce a new geometric characterization of the contact between parts moving in contact in terms of a conjugate *triplet* formed by the two shapes maintaining contact *and* their relative motion. We show that every triplet belongs to at least two classes of functionally equivalent shapes that may be uniquely represented by *maximal* conjugate triplets corresponding to the largest shapes that contain all other designs satisfying the same contact constraints. This characterization does not impose or require any specific parametrization of the design space of the higher pairs, and therefore is particularly effective in the conceptual design stage because it enables systematic exploration of an essentially unlimited space of solutions that can be performed through standard geometric computations.

1.3 Preliminaries

In this section, we briefly summarize the main concepts necessary for understanding the material in the rest of the paper. Motions are playing an important part in problems involving contact constraints because contact and relative motion are interdependent. However, motion is easier and more intuitive to specify, and the relative motion between two parts usually embodies the contact function of the two parts. A motion M is a one parameter family of transformations $M(t)$, where the parameter $t \in [0, 1]$. For the purposes of this work, "motions" and "transformations" are interchangeable and are commonly represented by matrices. Every point moving according to such a motion describes a trajectory, which is a continuous curve in the d dimensional space in which the motion occurs. Note that, both the motion of the point and its trajectory are "seen" differently from a fixed coordinate system and a moving coordinate system. For precise definitions, see appendix A or [10].

At the very least, contact between two moving parts enforces their non-interference and this can directly translate into containment constraints imposed on the two parts. If A and B are the two parts moving in contact and M is their relative motion, then A must be contained, throughout M, in the complement of the

[1] Such parametrization of motion, typically in terms of time, should not be confused with assumptions on the number of degrees of freedom.

set that represents B, otherwise interference would occur. By symmetry, B must be contained at all times in the complement of A.

The `unsweep` operation introduced in [9,10] returns the largest moving set of points remaining inside a given containing set and hence directly addresses the design problem of parts satisfying containment constraints. Informally, for a given object A and a motion M, `unsweep` returns all points, and hence the largest set of points, that would remain inside the stationary containing set A while the points would be moving according to M. In other words, `unsweep` returns all points that will not collide and interfere with the complement[2] of A. The `unsweep` operation has two equivalent definitions, one in terms of the trajectory test for a moving point, and the other is in terms of an infinite intersection of the moving set. Furthermore, `unsweep` is dual to the general `sweep` operation in a very precise mathematical sense. The formulation of the `unsweep` operation, the precise nature of its duality with `sweep`, and its computational properties are detailed in [10]. Here we only present the main concepts.

If set A and motion M are as before, and $\hat{M}$ denotes the inverted motion[3], then the dual of the `sweep` is defined as the infinite intersection operation:

$$\texttt{unsweep}(A, M) = \bigcap_{q \in \hat{M}} A^q \tag{1}$$

The second definition of `unsweep` can be given in terms of trajectories of moving points. If T_x is the trajectory of a moving point $x \in A$, then

$$\texttt{unsweep}(A, M) = \{x \mid T_x \subset A\} \tag{2}$$

Briefly, the principal properties of `unsweep` can be stated as follows:

1. $\texttt{unsweep}(A, M)$ is the largest set of points that remains inside A under M;
2. definition (2) can be used to test whether a given point is "in", "on" or "out" of the set $\texttt{unsweep}(A, M)$;
3. if X^c denotes the complement of a set X, then the relationship between the two dual operations is given by:

$$[\texttt{unsweep}(A^c, \hat{M})]^c = \texttt{sweep}(A, M) \tag{3}$$

   Intuitively, the duality (3) may be visualized as follows: instead of sweeping the object A according to M, one can `unsweep` the complement of A with the inverted motion $\hat{M}$ and then take the complement of the resulting set. But note that the inverted trajectory of a point under $\hat{M}$ is generally not the same as the trajectory of the point under motion M.

The first property implies that `unsweep` corresponds to a material removal operation. The second property indicates that `unsweep` can be computed effectively (see [10] for more details). The third property establishes the precise relationship between `unsweep` and general sweeps. In the next section, we will rely on these properties of `unsweep` to define precisely what it means for two moving parts to remain in contact.

1.4 Outline

Section 2 introduces the concept of conjugate triplets, and we show that every such triplet induces two maximal triplets containing the maximal shapes that are guaranteed to contain all other possible solutions to a given contact design problem. We also show in this section that every conjugate triplet defines one or more classes of functionally equivalent designs for the contact problem and that, for a given triplet, the two maximal triplets are unique. Section 3 illustrates the use of the proposed characterization and of the family of functionally equivalent parts in the design of an automotive hood latch and uses the example of a simple cam-follower mechanism to discuss the issues related to the loss of contact in a triplet. In the last section, we briefly discuss the implications of this work and suggest extensions towards defining other functional equivalence classes for moving parts aimed at providing a common framework for the design of moving mechanical parts.

[2] If we use here the regularized complement instead, then `unsweep` will contain all points that may contact but will never interfere with the regularized complement of A.

[3] If motion M is represented by the usual 4×4 matrix in homogeneous coordinates, the inverse $\hat{M}$ is the motion that is described, for every parameter value t, by the inverse of the matrix describing M at t. See appendix A for precise definitions.

2 Formulation

2.1 Conjugate Triplets

A key departure from the classical kinematic notion is that a kinematic "pair" involves in fact *three* (and *not* two) elements: two shapes and a motion. If we were to consider only the two moving shapes, they may or may not come in contact depending on how they actually move relative to each other: in general, there are many possible relative motions that would maintain contact of the two shapes as well as many that would not maintain contact. Specifying both shapes and the relative motion M completely specifies the structure and the kinematic behavior of the higher kinematic pair.[4] Thus, we define

Definition 1 *A* **conjugate triplet** $\langle A, B, M \rangle$ *consists of two objects A and B moving relative to each other according to a motion M and maintain contact with each other at all configurations $q \in M$.*

In what follows, a triplet will be denoted by Ψ, and if $\langle A, B, M \rangle$ is a conjugate triplet, we will refer to A and B as *conjugate* shapes or the pair associated with the triplet $\Psi = \langle A, B, M \rangle$. Without any loss of generality, we will also assume that M is the motion of A relative to B. Note that the notion of conjugate triplets remains valid for multiple contacts between A and B. If A or B have multiple simultaneous contacts with more than one object, then every such pair of objects will form a separate conjugate triplet.

There are several aspects of the conjugate triplets that may not be immediately apparent:

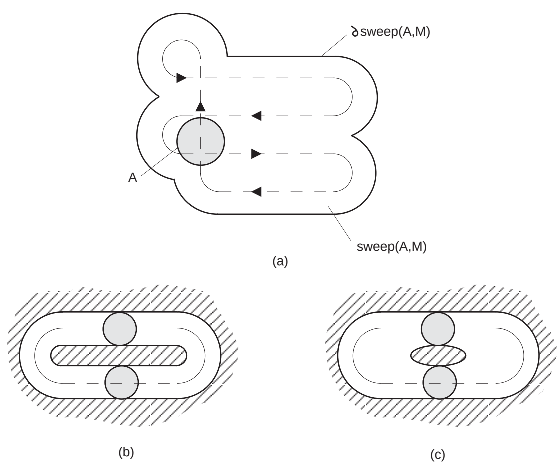


Fig. 2 A conjugate triplet might not exist for given object and motion and this is illustrated in (a). If one conjugate triplet exists (b), there is an infinite number of conjugate triplets: (c) shows one more possible conjugate triplet for the case illustrated in (b).

- A conjugate triplet may not exist for given shapes and motions. To illustrate such a case, we show in Figure 2(a) a disk whose center moves along the dotted line. It should be clear that there is no shape that remains in contact with the disk at all instances of the shown motion. Observe that in this case, the envelope (not shown) of the moving disk is self-intersecting.
- If a conjugate triplet does exist, then there are infinitely many such conjugate shapes for the same motion. We illustrate this case by showing two possible triplets in Figure 2(b) and 2(c). In both cases a disk is moving according to the same motion, but the object that remains in contact with the disk is different in each case: the object in Figure 2(b) is a superset of the object shown in Figure 2(c).

[4] In fact, another fundamental distinction between the higher and the lower pairs is that the motion of the lower pair is determined uniquely, apparently by the continuous symmetry (Lie) groups of rigid motions [18].

According to definition 1, both examples shown in Figures 2(b) and 2(c) are conjugate triplets, even if our intuition may suggest that there are some fundamental differences between the two. It should be also clear that there are infinitely many other conjugate triplets with the same motion M and infinitely many other shapes A and B that will maintain contact with each other during the relative motion. A typical situation in mechanical design is that one of the objects, say A, and a relative motion M are given and we want to design the second object B of the conjugate pair. But examples in Figure 2 illustrate that even in this case the problem appears to be ill-posed until we can answer at least some of the following questions:

- Given two such triplets, is it reasonable to say that they perform the same mechanical function?
- If they are functionally equivalent, what are precisely the corresponding equivalence classes and how can they be characterized?
- Among the infinitely many equivalent shapes and triplets, are some better than others for mechanical or computational purposes?
- Can we use the triplets to represent, compute, and synthesize higher pairs?

Answering these questions presumes that formal definitions of the above concepts, and particularly the notion of functional equivalence, have been formulated. Defining specific equivalence classes of mechanical parts should aim at providing functional criteria for determining whether two mechanical parts perform the same mechanical function. It should be clear that functional equivalence cannot be defined generally and uniquely, because it would imply complete and unique characterization of the corresponding mechanical function. Nevertheless, in section 2.2 we will define one such equivalence class of triplets for parts moving in contact, which explicitly represents the best/worst case scenario for problems involving contact constraints. One can envision, however, other classes of equivalence and we will briefly comment on these issues in section 4.

2.2 Positional Equivalence of Conjugate Triplets

We saw earlier that if a conjugate triplet exists for given A and M, there are infinitely many shapes B maintaining contact with A during M and therefore there are infinitely many triplets for the same motion. This observation allows to define a useful equivalence class of all such triplets. Informally, two triplets will be called *positionally* equivalent if they share the same set of configurations of their motions and the two sets of contact points corresponding to the two triplets are subsets of each other. In this section we will formalize these definitions.

Because the contact is fundamental for describing the mechanical functions of conjugate triplets, the definition of equivalence of conjugate triplets requires the introduction of a new concept, namely the *contact boundary* of conjugate shapes. Given a triplet Ψ, it is defined as follows:

Definition 2 *For a conjugate triplet $\Psi = \langle A, B, M\rangle$, the* **contact boundary** *of moving A is denoted by $\partial_c A$ and is the set of points of contact between A and B during M, or*

$$\partial_c A = \{x \in \partial A \mid x^q \in \partial B,\ q \in M\}$$

A similar definition can be stated for the contact boundary of B. Recall that B moves relative to A according to the inverted motion $\hat{M}$, and therefore

Definition 3 *For a conjugate triplet $\Psi = \langle A, B, M\rangle$, the* **contact boundary** *of B that moves in contact relative to A according to motion $\hat{M}$ is the set of points of contact between B and A throughout $\hat{M}$, or*

$$\partial_c B = \{x \in \partial B \mid x^p \in \partial A, p \in \hat{M}\}$$

Figure 3 shows a sphere A translating between two planes B so that the contact points N and S between the sphere and the planes move along the trajectories shown. The two trajectories represent the contact boundary of B, while the two points N and S represent the contact boundary of A. By definition, $\partial_c A \subset \partial A$ and $\partial_c B \subset \partial B$, and the two concepts are not redundant. On the other hand, the contact boundary may be defined from either one of the moving objects; for example, the contact boundary of B can be also defined with respect to motion M as

$$\partial_c B = \{x \in \partial B \mid x \in A^q, q \in M\}$$

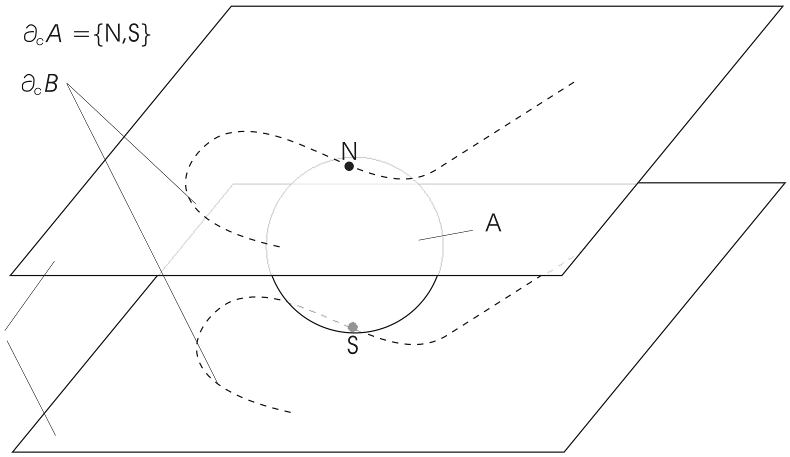


Fig. 3 The contact boundary $\partial_c B$ of B is the set of points belonging to the two trajectories lying in the two planes, while the contact boundary $\partial_c A$ of A is made of two discrete points, N and S.

The second ingredient of positional equivalence of triplets requires viewing every motion M as a set of configurations irrespectively of how this set may be parameterized by time t. This allows us to focus on existence of contact, while ignoring the differences in velocities, accelerations, or particular parameterizations of motions. Consider now two relative motions M_1 and M_2 in configuration space $\mathcal{C}$.

Definition 4 *The motions M_1 and M_2 are said to be* **configuration-equivalent**, *or* **c-equivalent** *for short, and we denote this by $M_1 \asymp M_2$, if the corresponding sets of relative configurations of M_1 and M_2 are equal, or $\bigcup_i M_1(t_i) = \bigcup_j M_2(t_j), t_i, t_j \in [0, 1]$ in the configuration space $\mathcal{C}$.*

The above definition states that the two motions "cover" the same relative configurations in $\mathcal{C}$. Being a one parameter family of configurations, a motion has an inherent sequence of configurations built in and indexed by the parameter of the motion. Although the two motions are continuous by assumption, the above definition does not restrict the order in which the relative configurations are occupied, as long as the two sets in the configuration space are the same. In other words, two motions are c-equivalent if any given point sweeps the same trajectory according to both motions. An example of an object A moving according to two c-equivalent motions is shown in Figure 4. Note that the c-equivalence of motions is an important example of an equivalence class, but which does not sufficiently characterize our contact problem.

The notion of equivalence inherently implies a comparison, or a condition, that must be satisfied. With all the above definitions we can now state the condition for the positional equivalence. It demands the inclusion relationship to hold between the contact boundaries *and* the equality between the sets of configurations. More precisely:

Definition 5 *We say that a triplet $\Psi_1 = \langle A_1, B_1, M_1 \rangle$ is a* **restriction** *of another triplet $\Psi_2 = \langle A_2, B_2, M_2 \rangle$, and we denote this by $\Psi_1 \leq \Psi_2$, if the following three conditions are satisfied:*

- *$M_1 \asymp M_2$ in configuration space $\mathcal{C}$,*
- *$\partial_c A_1 \subseteq \partial_c A_2$,*
- *$\partial_c B_1 \subseteq \partial_c B_2$.*

The restriction is essentially a "less than or equal" relationship. To define an equivalence class, we only need to provide the largest element in the class. Thus, the positional equivalence of conjugate triplets induced by some largest known triplet Ψ is simply defined as follows:

Definition 6 *Two triplets Ψ_1 and Ψ_2 are* **positionally equivalent** *if they are restrictions of some conjugate triplet Ψ.*

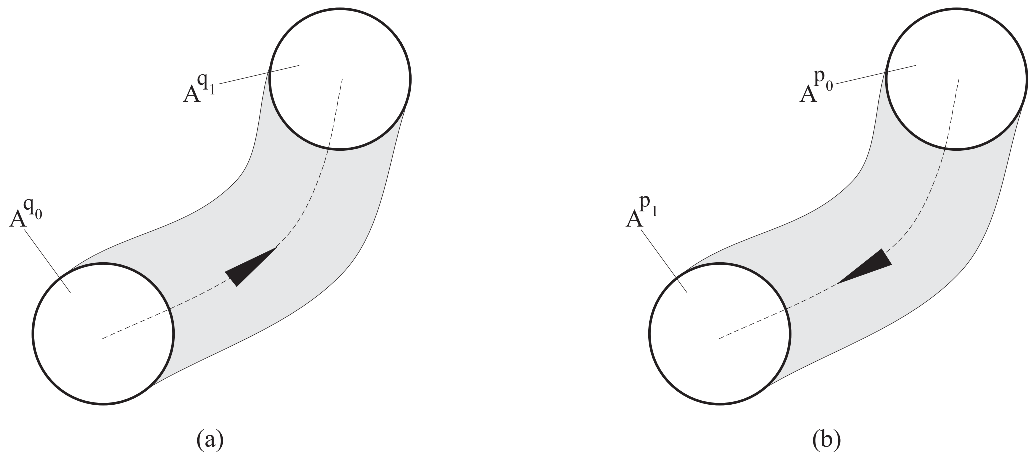


Fig. 4 The same object moving according to two c-equivalent motions occupies the same set of configurations although not necessarily in the same sequence. Figure (a) shows the object A moving from configuration q_0 to q_1 of M_1 while Figure (b) shows the same A moving from $p_0 = q_1$ to $p_1 = q_0$ of M_2.

It is easy to show that the positional equivalence with respect to some *fixed* triplet Ψ is indeed an equivalence relation, in the same sense that all numbers less than some fixed number X are equivalent to each other. Similarly, positional equivalence with respect to Ψ is both symmetric and transitive.

Thus, *any* given triplet immediately induces an equivalence class of triplets that are its restrictions. For example, Figure 5 shows an example of two triplets in which $\Psi_2 \leq \Psi_1$ and hence they are also positionally equivalent with respect to Ψ_2. Both triplets contain the same set A and the same relative motion M (a linear translation), but $B_1 \neq B_2$. Observe that the contact between A and B_1 on one hand and A and B_2 on the other hand is made along different sets of points. More specifically, $\partial_c B_2 \subset \partial_c B_1$, although set B_2 is larger than B_1. Based on the above definitions, the triplets depicted in Figure 2(b)-(c) are also positionally equivalent.

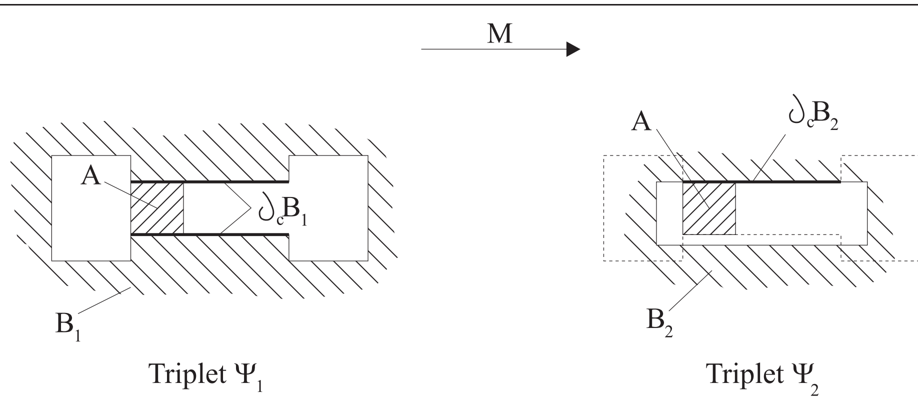


Fig. 5 Two triplets Ψ_1 and Ψ_2 such that $\Psi_2 \leq \Psi_1$ are also positionally equivalent.

Given two triplets with c-equivalent motions, we can test if one of them is a restriction of the other by checking whether the two conditions contact boundaries in the Definition 5 hold. But even if these conditions do not hold, it is still possible that the two triplets belong to the same positional equivalence class induced by some other "larger" triplet. To be practically useful, such a larger triplet should intuitively correspond to the same mechanical function as performed by the given triplet(s). We now show that any given triplet naturally belongs to two unique distinct equivalence classes induced by two *maximal* triplets; the latter correspond precisely to the tasks of designing the objects A and B of the conjugate pair.

2.3 The Maximal Conjugate Triplets

Given a triplet $\Psi = \langle A, B, M \rangle$, let us fix the relative motion M and conceptually grow the objects B and A while preserving the intended conjugate motion. Without additional restrictions, growing is not a deterministic process, but it should be clear that growing objects may only produce additional contacts between A and B, while it can never eliminate the contacts that are already present. Thus, the given triplet Ψ is guaranteed to be in the equivalence class of the grown triplet. Because we have a choice of which object to grow first, this conceptual process is not unique, but clearly cannot be continued indefinitely.

Without loss of generality, let us assume that we are more interested in the possible shapes for B, perhaps because A is already sufficiently constrained. Growing first B as much as we can, gives the largest possible object B^A_{max} that moves in contact with A according to M. Now, we can also consider growing A, if the contact between A and B^A_{max} allows it, obtaining a new set A^A_{max}. Formally, we can identify a special conjugate triplet $\Psi^A = \langle A^A_{max}, B^A_{max}, M \rangle$ with conjugate shapes defined by:

$$\begin{aligned} B^A_{max} &= k[(\texttt{sweep}(A, M))^c] \\ A^A_{max} &= k[(\texttt{unsweep}((B^A_{max})^c, M))] \end{aligned} \tag{4}$$

where X^c denotes the standard set complement of a set X, and kX is the closure of X. The properties of `sweep` and `unsweep` operations[5] imply that for a given Ψ, both B^A_{max} and A^A_{max} are well defined, and in this sense Ψ^A is unique.

By definition, the complement of the set swept by a moving object contains all points that will not come in contact with the moving object. In addition, the boundary of this set contains all points that will come in contact with the moving object. Hence, the set B^A_{max} defined in equation (4) is the largest set that will come in contact without interfering with the moving A of Ψ and must contain all other sets B that will come in contact with A during M. We note that $A \subset A^A_{max}$, based on the properties of the `unsweep` operation discussed in [10]. In other words, A^A_{max} is the largest A that would generate the largest B during the given M. Figure 6(b) shows the resulting maximal triplet, starting with the triplet in Figure 6(a).

The above process and construction are clearly order-dependent and asymmetric with respect to A and B. If A moves relative to B according to M, then B moves relative to A according to the inverted motion $\hat{M}$. Therefore, there exists another unique conjugate triplet $\Psi^B = \langle B^B_{max}, A^B_{max}, \hat{M} \rangle$ defined as in (4) with A and B interchanged, with the maximal set A^B_{max} that will come in contact with B while B moves relative to A^B_{max} according to $\hat{M}$. Figure 6(c) shows an example of the second maximal triplets Ψ^B in which A and B are two rectangular shaped sets depicted in (a) and motion M is a translation along the indicated line. Because A and B do not lose contact during M, $\langle A, B, M \rangle$ is a conjugate triplet.

The importance of the maximal triplets to the shape design of higher pairs lies in the fact that the maximal triplets are made up from the largest (and hence unique) shapes that maintain the specified motion and contact. As is clearly seen in Figures 6(b) and (c), the maximal triplets Ψ^A and Ψ^B are composed from very different parts A and B, even though they involve c-equivalent relative motions. The shape of a given object A is altered but is largely preserved in Figure (b), while the original shape of B is still visible in Figure (c). This should not be surprising: each maximal triplet defines a class of all shapes that can be considered functionally equivalent in a precise sense. If the initial object A is given, then the function of the triplet is to maintain a moving contact with the given A, and the designer should focus on the properties of the maximal triplet Ψ^A. Changes to A are permitted *only* to the extent that they do not affect the shape of the largest conjugate shape B, because this would effectively lead to a new design problem. By symmetry, the maximal triplet Ψ^B should be used when designing the shape of A to move in contact with a partially known object B. The maximal triplets provide powerful means for systematic comparison, analysis, and modification, and optimization of the positionally equivalent designs. Given a conjugate triplet, the two maximal triplets may be computed using the usual geometric modeling techniques for computing sweep and

[5] Observe that, by duality of `sweep` and `unsweep` (3), definitions (4) can be rewritten in terms of the corresponding dual operations.

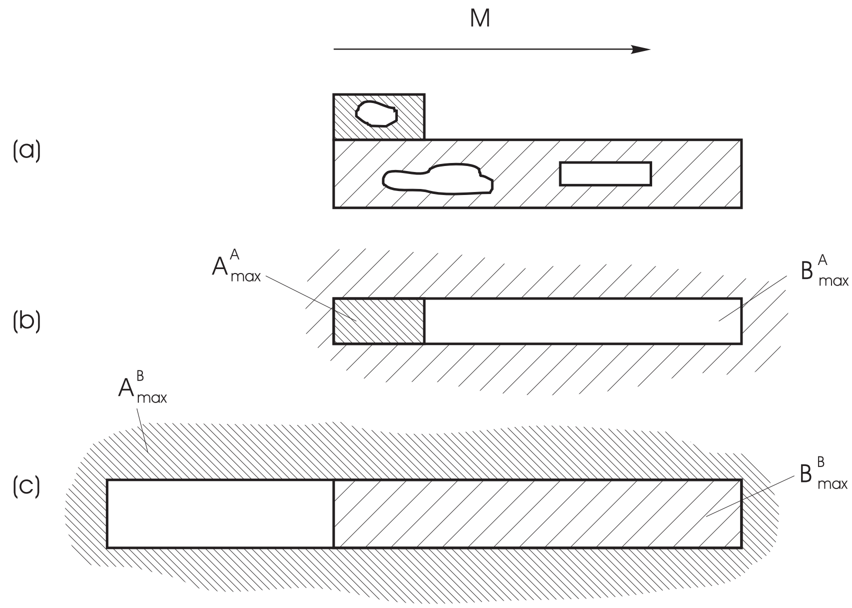


Fig. 6 Given a conjugate triplet $\langle A, B, M\rangle$ (a), the triplets Ψ^A and Ψ^B are shown in (b) and (c). Note that the holes in A and B disappeared in the corresponding maximal triplets.

unsweep operations. This effectively gives a computational procedure for deciding the functional equivalence of two conjugate triplets, and for generating new triplets in the same class [11]. Recall that if only one of the conjugate objects, say A, and motion are known, then the conjugate triplet may not exist. This fact would be immediately revealed through non-existence of Ψ_A defined by equations (4) because the contact boundary of the set B^A_{max} will be empty. Specific example applications of maximal triplets and the induced equivalence classes are discussed in the next section and more extensively in [11].

3 Designing Contacts Through Equivalence Classes

In a typical design problem, partially shaped conjugate objects A and B must move in contact according to some prescribed relative motion. The maximal triplet Ψ^A (Ψ^B) contains all the sets that would remain in contact with A (B) during M. Such classes of equivalence of mechanical parts lay the ground for the development of design space exploration techniques to search the design space for solutions that have functional characteristics similar (in a predefined sense) with those of an existing solution. Choosing one solution among the infinitely many possibilities must take into consideration other factors that are functionally significant, such as the applied loads, and additional constraints on the final part such as containment, strength, manufacturability and so on. We now demonstrate the usefulness of this approach on two specific applications: design of automotive secondary latch and study of a typical cam-follower mechanism.

3.1 An Automotive Secondary Hood Latch

Figures 7 show a secondary hood latch, which is part of a typical hood latch assembly of an automobile. Broadly, the functionality of a hood latch assembly is to engage and retain a (generally cylindrical) striker *and* to prevent it from accidental release. In this example the striker is attached to the hood of the automobile. We focus on the contact function of the latch, since engaging the striker and avoiding its accidental release is accomplished by the contact between the striker and the latch. The design of latch is also affected by other constraints, including strength, containment, and manufacturability, as discussed in [8, 10].

With reference to Figures 7(a-e), during its downward motion, a vertically moving striker forces the secondary latch to rotate and latch it. While the striker continues to move vertically down, the secondary latch is brought back to its original position by an attached reaction spring; the primary latch, not shown, closes down and holds the striker fixed. The release of the striker from the hood latch assembly is accomplished in two phases: first, manually applying external forces indirectly to the primary latch from the cabin, and then directly to the secondary latch – so that the secondary latch rotates and the hood can be lifted. A spring (not shown) attached to the secondary latch provides the reaction force needed to insure the return to its original position, but the latch must continue to function even if the spring breaks. The secondary latch must also prevent the upward motion of the striker (and therefore of the hood) when the primary latch fails to engage the striker. In addition, the secondary latch has to remain inside a specified containing set during its motion to avoid interference with neighboring parts.

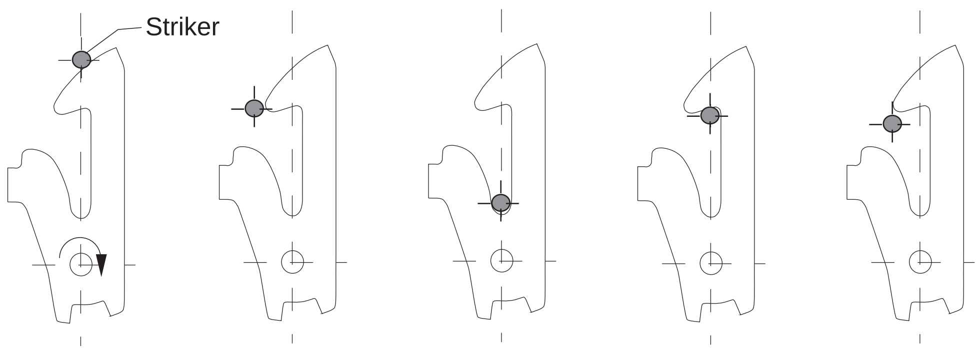


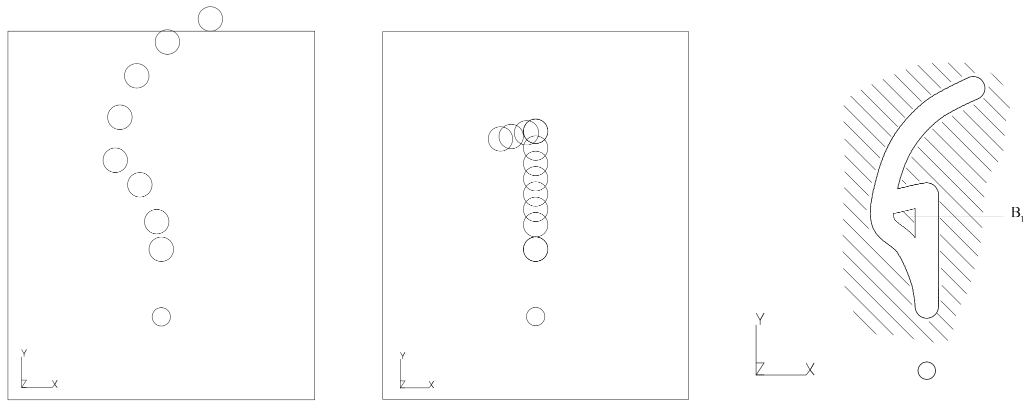
Fig. 7 A secondary hood latch must engage and retain a vertically moving striker.

This latch illustrates an example of a part that is often redesigned to accommodate the new car models and/or changing requirements [8]. Since the geometry of the latch does not contain in itself the information on functionality of the latch or the reasons that led the designer to choose this particular geometry, the redesign process often requires quesswork or designing the part from scratch. In turn, this leads to costly "generate and test" design procedures that may not directly capture the original design intent. Formulating the design problem as a conjugate triplet, we are given the shape of the cylindrical striker A and partial information about motion M, and we need to determine the shape of the object B that moves in contact with A. Computing the corresponding maximal triplet Ψ_A solves the design problem because it represents the class of all functionally equivalent latches B.

In the initial stages of the design of such a latch, the designer does not have an analytical representation for the relative motion M, but rather has a feeling for how the relative motion between the striker and the latch may look like. We assume that such a partial understanding of the relative motion is properly expressed by a partial description of the motion through a set of relative configurations of the striker and the latch to be designed. Such a set of relative configurations is shown in Figure 8 where the relative motion of the striker was separated into its "downward" motion and its "upward" motion corresponding to the relative motions for the engagement and the release of the striker by the secondary hood latch (see also Figure 7).

The relative configurations were interpolated in this example using the algorithm described in [7], which resulted in a continuous description of the motion of the striker relative to the latch. Different design cases may require different interpolation algorithms producing motions with dissimilar properties: in some cases the smoothness and continuity of the resulting motion can be essential, while in others, the simplicity of the resulting shape may be more important.

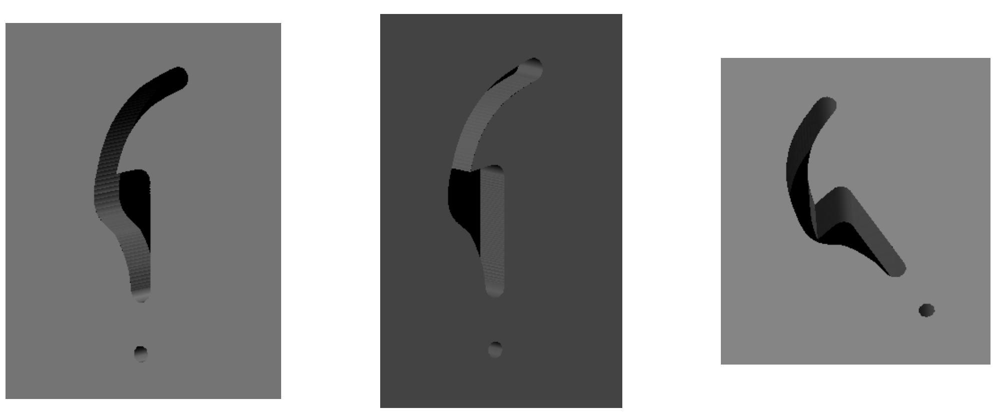
Once the motion M is known, it is straightforward to compute the maximal shape B^A_{max}, which is shown in Figure 8(c). If we require the latch to be a single part, it should be a connected set, and we must be able to maintain the contact with the given striker A using only one of the connected components of B^A_{max}. The straightforward analysis in this case indicates that the connected component B_1 shown in Figure 8(c) is not



(a) (b) (c)

Fig. 8 The discrete relative configurations of the striker A relative to the latch to be designed: the motion of the striker was separated into its "downward " motion (a) and its "upward" motion (b) because they correspond to different functions of the latch. Figure (c) shows the top view of the B^A_{max} forming the maximal triplet $\langle A, B^A_{max}, M\rangle$.

required to maintain the contact. The positionally equivalent connected shape represents the largest latch and is shown in Figure 9. It contains all other positionally equivalent latches satisfying the given contact function, and checking whether new geometries of the latch satisfy the contact constraints reduce to relatively simple geometric computations.



(a) (b) (c)

Fig. 9 The largest connected latch, which includes all other positionally equivalent shapes satisfying the given contact function.

Refining the geometry of this largest latch satisfying the contact function must *also* take into account the applied forces, the containment constraints, the strength of the latch, the manufacturing constraints and so on. We will return briefly to these issues in section 4. Under all conditions, any final latch must form a triplet with the striker and their relative motion that is positionally equivalent with the maximal triplet shown in Figure 9.

3.2 Loss of contact in Cam-Follower Mechanism

The design of the cam-follower mechanism is typically preceded by a choice of the follower and of the relative motion (typically some piecewise polynomial function) between the follower and the cam. Without attempting a thorough literature review, the existing design methods for the cam shape can be grouped into graphical methods [4, 21], techniques based on envelope theory [6] or methods employing the screw theory [22]. In what follows, we assume that both the follower A and the relative motion M are known quantities. As the follower moves through space according to M, it is possible that boundary of A traverses the space in a manner that causes *local* loss of contact, sometimes referred to as 'undercutting.' We illustrate this situation in Figure 10 which shows the profile of an internal cam such as those used, for example, in rotary pumps [16] or rotary engines [17]. The functionality of such mechanisms requires the boundary of the follower to trace the outer rather than the inner profile of the cam, and the forces applied on the mechanism are ensuring that this contact is maintained. The conditions for detecting such a loss of contact usually assume particular parameterizations and require fairly complex differential analysis [22, 15]. However, such loss of contact does not necessarily imply loss of mechanical function, as is readily revealed by the analysis of the corresponding maximal triplet.

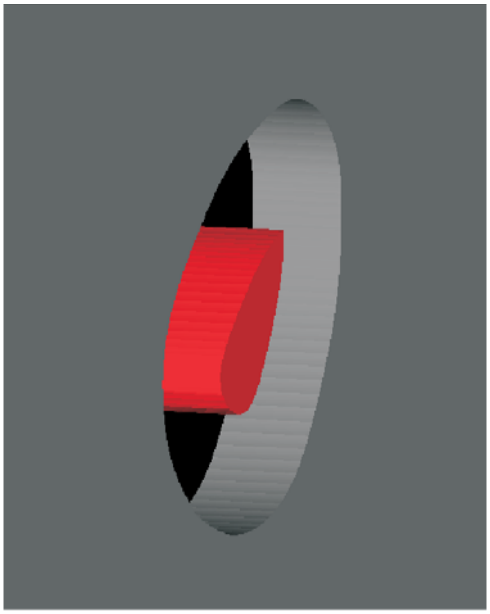
Computing the largest possible cam B^A_{max} allows to form the corresponding maximal triplet $\langle A, B^A_{max}, M\rangle$. Once again, this maximal triplet contains all other designs of cam-follower mechanisms. The local loss of contact may appear in the maximal triplet as well, but this does not necessarily mean that the triplet cannot be used for designing the functional kinematic pairs. As long as *some* contact is maintained at every relative configuration of the triplet, it may be possible to continuously apply the required contact forces on the moving objects. The final determination requires analysis of applied forces, and in this particular case the mechanism shown in Figure 10 is indeed functional. The loads applied on A (not shown) are such that the outer profile (the boundary of the unbounded region in Figure 10) of the cam maintains the contact with the follower A, although undercutting does occur near the inner surfaces of cam. As long as the follower A maintains contact with the outer surfaces of the cam B, $\langle A, B, M\rangle$ is a perfectly functional maximal triplet. On the other hand, if this triplet were not acceptable, no other functional mechanism can exist with the same geometry of follower A and motion M. Similarly, total loss of contact at any relative configuration of the follower and the cam would imply that the maximal triplet does not exist and no functional design is possible.

In addition to demonstrating the utility of the proposed approach, this example also shows that proper determination of the functional boundary of B requires taking into account the loads applied on the moving mechanism. We note that the similar conditions could be, in principle, applied to determining the functional profile of A.

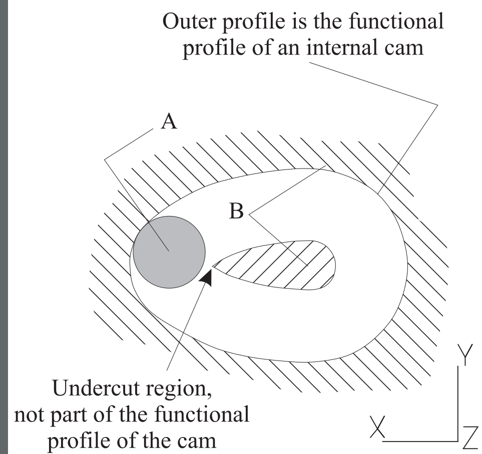
4 Conclusions

Classes of equivalence of mechanical parts promote systematic explorations of the space of design solutions for a given design problem. More generally, these classes transform an otherwise untractable problem into a tractable one because one can represent, in principle, infinitely many design solutions in terms of finitely many representative members of the class. Positional equivalence of triplets discussed in section 2.2 defines one such class of design solutions for the contact problem, solutions which, for the same set A, share the same set of configurations of their motions and set of contact points. We have also shown that the maximal triplets provide a new geometric characterization of the contact problem involving two parts moving relative to each other. Every such triplet defines a class of functionally similar designs all of which perform the same contact function. This class of designs can be uniquely represented in terms of two maximal triplets corresponding to the two maximal shapes that are guaranteed to contain all other design solution of the contact problem.

A design scenario, for example in the case of the latch discussed in section 3.1, can start with a known shape of the moving object (in this particular case the striker) and with partial or full information about the motion and external loads applied on the mechanism. Generating the maximal triplet Ψ_A, which is the representative member of the positional equivalence class defined in this paper, and to which the latch to be designed belongs, requires that the complete relative motion is known. This may require a motion interpolation step as discussed in section 3.1 or in more detail in [11]. The maximal triplet defines the



(a)



(b)

Fig. 10 The loads applied on A are so that the outer profile of the cam is the one that is functional, although undercutting of A does occur for the inner profile.

"boundary" of the design space for the design problem. This means that one can effectively start exploring the design space to determine a multitude of shapes that satisfy the additional constraints of the problem, such as loads, spatial containment, strength and so on. This scenario makes design process systematic and allows keeping keep track of all the design decisions that are invaluable in the subsequent production stages or re-designs of the part.

It is clear that there are other classes of equivalence that may better reflect the functionality of parts moving in contact; by definition, any such equivalence class is a *subclass* of the positionally equivalent class defined in this paper. For example, we argued that selected functional designs should also take into consideration the external loads applied to the triplet. The corresponding functional subclasses are studied in [11]. Other refinements of the maximal shapes should account for additional design constraints such as strength, manufacturing, and so on. Importantly, any part satisfying the same contact function must be part of a triplet which is positionally equivalent with the corresponding maximal triplet.

Due to limited space, we chose not to discuss in this paper many traditional tools for design and analysis of higher kinematic pairs that rely on graphical methods [4,21], on techniques based on screw theory [6, 5], or the theory of envelopes [22,15], and more recently on configuration space computations [12,13,3]. What is relevant in the context of this paper is that all these methods provide incomplete parametrizations of the space of design solutions, which necessarily limits the space of available solutions and makes their generalization to handle problems from other parametric families difficult. Since our characterization does not impose any specific parametrization of the design space, it reflects an essentially unlimited space of design solutions, thus being particularly effective in the conceptual design stage of moving mechanical parts. Furthermore, the explorations of the design space can be performed using the standard geometric modeling tools.

To conclude, any systematic mapping of the functional requirements of a mechanical part into a shape or form that satisfies these requirements must rely on an appropriate characterization of the design space of solutions. This work represents a first step towards that goal by defining classes of functionally equivalent moving mechanical parts.

Acknowledgments

This work is supported in part by General Motors Corporation, and by the National Science Foundation grants DMI-9502728 and DMI-9522806. The examples were created using the Parasolid modeling system, courtesy of UGS Corporation. Vadim Shapiro gratefully acknowledges the support of Dipartamento di Informatica e Automazione, Universita Roma Tre, during his sabbatical leave when this article was written.

A Appendix: Motions and Transformations

Consider a set of points A with its own coordinate system $\mathcal{F}_A$ moving in a d-dimensional Euclidean space $\mathcal{W}$ with respect to some global fixed coordinate system $\mathcal{F}_\mathcal{W}$. A configuration of an arbitrary object is a specification of the position of every point in this object relative to a reference frame [1]. Therefore, a configuration q of A is a specification of the position $\mathcal{T}$ and orientation Θ of $\mathcal{F}_A$ relative to $\mathcal{F}_\mathcal{W}$. The configuration space $\mathcal{C}$ of A is the space of all configurations q of A. Mathematical properties of such a configuration space are well understood and they are extensively discussed in [14].

A motion M is a one parameter family of transformations $M(t)$, where the parameter $t \in [0,1]$. We normalize all other intervals $[a,b]$ to $[0,1]$, so that the results presented here remain valid when $t \in [a,b],\ 0 < a < b$. For the purposes of this work, "motions" and "transformations" are interchangeable and are commonly represented by matrices.

A rigid body motion in a d-dimensional space is determined by $\frac{d(d+1)}{2}$ independent degrees of freedom, as a path in the configuration space $\mathcal{C}$. Each instantaneous transformation $M(a)$, for any $a \in [0,1]$, has a unique inverse $\hat{M}(a)$ such that $x = \hat{M}(a)[M(a)x]$. For a range of values of $t \in [0,1]$, the *inverted* motion $\hat{M}(t)$ of a motion $M(t)$, is the inverse of $M(t)$ for every instance of t.

Each point x of the moving object A that moves according to M describes a trajectory

$$T_x = \{x^q, q \in M\} \tag{5}$$

where x^q denotes point x at configuration q. Consequently, in what follows we will denote set A at configuration q by A^q. From the coordinate system $\mathcal{F}_A$ attached to the moving object A, point x will appear to be moving along the *inverted* trajectory

$$\hat{T}_x = \{x^p, p \in \hat{M}\} \tag{6}$$

Importantly, the relationship between the trajectory and the inverted trajectory of point x is generally not simple. A more detailed discussion can be found in [10].

In the above discussion, motion M can be considered as a special case of a more general *relative* motion with the coordinate system $\mathcal{F}_\mathcal{W}$ fixed in $\mathcal{W}$. If $\mathcal{F}_\mathcal{W}$ were not fixed, then motion M would represent the relative motion between $\mathcal{F}_A$ and $\mathcal{F}_\mathcal{W}$. Let M_A and M_W be the *absolute* motions of $\mathcal{F}_A$ and $\mathcal{F}_\mathcal{W}$ relative to a fixed coordinate system in $\mathcal{W}$. The relative motion between $\mathcal{F}_A$ and $\mathcal{F}_\mathcal{W}$ can be expressed in any coordinate system defined in the same space, but usually it is convenient to have it expressed in one of the two moving coordinate systems. Then, the relative motion is expressed in $\mathcal{F}_A$ by

$$M_{A/W} = {M_W}^{-1} M_A \tag{7}$$

and, consequently, in $\mathcal{F}_\mathcal{W}$ by

$$M_{W/A} = {M_A}^{-1} M_W. \tag{8}$$

Equation (7) expresses the motion of $\mathcal{F}_A$ observed from $\mathcal{F}_\mathcal{W}$, while equation (8) expresses the motion of $\mathcal{F}_\mathcal{W}$ as observed from $\mathcal{F}_A$. Note that both equations (7) and (8) express the *same* physical relative motion between $\mathcal{F}_A$ and $\mathcal{F}_\mathcal{W}$, but in different coordinate systems.

References

1. Vladimir I. Arnold. *Mathematical Methods of Classical Mechanics*. Springer-Verlag, New York, 1978.
2. Ivan I. Artobolevsky. *Mechanisms in Modern Engineering Design*. MIR publishers, Moscow, 1975.
3. M.E. Caine. *The Design of Shape From Motion Constraints*. PhD thesis, Massachusetts Institute of Technology, 1993.
4. Fan Y. Chen. *Mechanics and Design of Cam Mechanisms*. Pergamon Press, 1982.
5. David B. Dooner and Ali A. Seireg. *The Kinematic Geometry of Gearing*. John Wiley & Sons Inc., 1995.
6. Max A. Gónzales-Palacios and Jorge Angeles. *Cam Synthesis*. Kluwer Academic Publishers, 1993.
7. T. Horsch and B. Jüttler. Cartesian spline interpolation for industrial robots. *Computer Aided Design*, 30(3):207–224, 1998.
8. H. Ilieş and V. Shapiro. An approach to systematic part design. In *Proceedings of the 5th IFIP WG5.2 Workshop on Geometric Modeling in CAD*, pages 383–392, may 1996.
9. H. Ilieş and V. Shapiro. `UNSWEEP`: Formulation and computational properties. In *Proceedings of the fourth ACM Symposium on Solid Modeling and applications: Solid Modeling '97*, Atlanta, Georgia, May 1997.
10. H. Ilieş and V. Shapiro. The dual of sweep. *Computer Aided Design*, 31(3):185–201, March 1999.
11. Horea Ilieş. *On Shaping Moving Mechanical Parts*. PhD thesis, University of Wisconsin, Madison, 2000.
12. L. Joskowicz and S. Addanki. Innovative shape design: A configuration space approach. Technical Report 356, Courant Institute of Mathematical Sciences, New York University, March 1988.
13. L. Joskowicz and E. Sacks. Computer-aided mechanical assembly design using configuration spaces. Technical Report Computer Science Technical Report 97-001, Purdue University, 1997.
14. Jean-Claude Latombe. *Robot Motion Planning*. Kluwer Academic Publishers, Boston / Dordrecht / London, 1991.
15. Feyodor L. Litvin. *Gear Geometry and Applied Theory*. Prentice-Hall, 1994.
16. Lev Nelik. *Centrifugal and rotary pumps: fundamentals with applications*. CRC Press, 1999.
17. Jan P. Norbye. *The Wankel engine design, development, applications*. Philadelphia Chilton Book Co., 1971.
18. M. O'Connor, V. Srinivasan, and A. Jones. Connected Lie and symmetry subgroups of the rigid motions: Foundations and classification. Research Report RC 20512, IBM, July 1996.
19. Jack Phillips. *Freedom in Machinery*. Cambridge University Press, 1984.
20. Franz Reuleaux. *The Kinematics of Machinery*. Macmillan and Company, 1876.
21. Joseph E. Shigley and John J. Jr. Uicker. *Theory of Machines and Mechanisms*. McGraw-Hill, 1995. 2nd ed.
22. D.M. Tsay and H.M. Wei. A general approach to the determination of planar and spatial cam profiles. *Journal of Mechanical Design*, 118:259–265, June 1996.